MATHEMATICAL MODELLING OF INFECTIOUS DISEASES: ENVIRONMENTAL AND DEMOGRAPHIC EFFECTS

A Thesis Submitted

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

by

MINI GHOSH

to the

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

September, 2000

Dedicated

To

My Parents

CERTIFICATE

It is certified that the work contained in the thesis entitled " **Mathematical Modelling of Infectious Diseases: Environmental and Demographic Effects**", by **Mini Ghosh (Roll No: 9410865)**, has been carried out under our supervision and that this work has not been submitted elsewhere for a degree.

Peeyush Chandra
Professor
Dept. of Mathematics
I.I.T.Kanpur

Prawal Sinha
Professor
Dept. of Mathematics
I.I.T. Kanpur

September, 2000.

Acknowledgement

I express my sincere gratitude to Professor Peeyush Chandra and Professor Prawal Sinha, for their guidance and support over the years. I am extremely grateful to them for introducing me to epidemiological modelling.

Special thanks are due to Professor J. B. Shukla who had always time to listen and to give me his valuable comments and suggestions. I thank him for his help.

I am indebted to all the faculty members of the department of Mathematics, IIT Kanpur, particularly to Professor V. Raghavendra for helping me in various matters.

I thank the staff members of the Department of Mathematics, IIT Kanpur for their help during this period.

I extend my thanks to Professors at S. M. College, Bhagalpur, Professor Aruna Roy and Professor Mili Ganguly for encouraging and inspiring me to pursue higher study in Mathematics.

I owe my thanks to all my colleagues especially Bishwaranjan, Dipak, Kailash, Manish, Muthu, Parasar, S. B. Rao and Sajid for their enthusiastic help throughout my stay at IIT Kanpur. I acknowledge Amrapali, Chaitali, Kusum, Malabika, Monideepa, Reeta and Shruti for helping me in various matters. My friends Sushmita and Swagata have been always there whenever I needed them for help and advice. I remain indebted for their love, encouragement and support.

At last, but by no means the least, I wish to express my deep gratitude to my parents, brothers Prashant and Rajiv and sister Mili, whose love and support have been an invaluable source of inspiration to me. Their faith in my abilities and their encouragement have been instrumental throughout this period.

Mini Ghosh

Synopsis

It is well known that the spread of various infectious diseases in human populations, is one of the most serious problems that modern society faces today as it causes mortality of millions of people, and costs enormous amount of money in health care and control of diseases. In general the spread of an infectious disease is governed by factors such as : (i) the density of the human population, (ii) the density of the carrier, bacteria or vector population, (iii) various environmental factors such as household discharges, rain, temperature, humidity etc, (iv) ecological factors such as vegetation and biomass, (v) geographical factors. It is noted here that the study of the spread of infectious diseases using mathematical models by considering some of these factors has not been done, particularly when the density of the carrier, bacteria and/or vector populations are variables. For example, household discharges may be very conducive to the growth of the carrier population, such as flies, cockroaches as well as the bacterial population and the vector population (such as mosquitoes). This increase helps in enhancing the contact rate of infectives with susceptibles, leading to fast spread of the disease.

Keeping the above in view, we have modeled the spread of infectious diseases by using simple mass action incidence. The effects of indirect contact with carriers, bacteria or vectors have been included in the models proposed. Two types of human demography have been considered, (i) constant immigration and (ii) logistic growth. The thesis consists of seven chapters and is divided into two parts. The first part deals with modelling infectious diseases in a single population whereas the second part deals with the spread of infectious diseases into two populations living in two different environmental conditions.

The first chapter provides the introduction regarding the dynamics of infectious diseases including a brief literature survey relevant to modelling and analysis of infectious diseases, so that the work done in the thesis can be seen in its proper perspective.

In the second chapter, we study the effects of the following factors on the spread of a carrier-dependent infectious disease using SIS mathematical models:

(i) variable carrier population, the density of which follows a logistic model and increases further due to household discharges in the habitat,

(ii) human demography.

The following two types of demographic factors are considered in the analysis :

(i) a population with constant immigration and death rate,

(ii) a population with logistic growth.

It is shown that the endemic equilibrium is globally stable when the rate of cumulative environmental discharges, conducive to the growth of carrier population, is a constant. When the rate is a function of human population density, the endemic equilibria is also locally and globally stable but under certain conditions. The global stability is also demonstrated by computer simulation. It is shown that if the human population increases due to demographic changes, the spread of the infectious disease increases. It is further noted from the analysis that the spread of the infectious disease increases as the growth rate of the carrier population (caused by conducive environmental discharges due to human sources) increases.

In the third chapter, we have considered a bacterial disease model, which is very similar to the models of Chapter 2 involving carriers. Here the only difference is that bacteria is also released by infectives continuously into the habitats and its growth rate increases not only because of household discharges but also by continuous release from infectives. Here the SIS models for bacterial infectious diseases, (like T.B., typhoid, etc.), which are caused by direct contact of susceptibles with infectives and also by bacteria, are proposed and analyzed quantitatively as well as by simulation. It is found again that the endemic equilibria is globally stable when the rate of cumulative environmental discharges conducive to the growth of bacterial population is a constant. But when the rate of discharge is a function of human population density, the endemic equilibrium is locally and globally stable only under certain conditions. It is further shown that under various demographic situations mentioned above, the infective population increases with the increase of household discharges and the spread of the disease increases and it becomes more endemic.

In Chapters 4 and 5, malaria models without and with reservoir have been proposed and analyzed by considering,
(i) the effect of cumulative discharges from household sources, and (ii) the effects of demographics of the human and mosquito populations.
The cases of constant as well as variable discharge rates are considered. The threshold parameter for spread of the diseases is derived in each case. If this threshold is greater than one, the corresponding nontrivial equilibrium is always feasible and is locally asymptotically stable to small perturbations. It is also shown by analysis and simulation that this equilibrium is globally stable under some conditions. The effect of demographic increase in the human population is found to increase the spread of malaria. It is observed that as the cumulative environmental discharges increase, the mosquito population increases, leading to fast spread of malaria.

In the Chapter 6, a model is considered to study the effect of immigration of population from an environmentally degraded habitat to an environmentally cleaner habitat on the spread of an infectious disease. Constant as well as variable household discharge rates are considered. In each case the conditions for existence of endemic equilibrium points have been obtained. It has been shown that under certain conditions, these equilibria are locally stable indicating that the disease may become endemic. Further by simulation, it is observed that the endemic equilibrium points are in fact globally stable under local stability conditions. It is concluded that as the migration rate increases, the total and the infective population densities of the region with cleaner environmental conditions increase, where as the total and the infective population densities of the region of degraded environmental condition decrease as expected.

Lastly in Chapter 7 a mathematical model is proposed by keeping in view that a poor population living in an environmentally degraded habitat spreads the disease by interacting with the rich population by working in their habitat as service providers (without migration). The model is presented under similar environmental conditions as in Chapter 6 and as in the previous chapters, here also linear and nonlinear analysis of the equilibria

have been studied for various cases. It is seen that as the interaction between rich and poor classes increases, the disease spreads more rapidly in the rich class as expected.

It is suggested that it is in the interest of the rich class to not only clean their own environment but also to clean the environment of the districts in their neighborhood, where poor people live.

Contents

List of Figures

Chapter 1

General Introduction

1.1 Dynamics of Epidemic Diseases

Epidemiology is a discipline which deals with the study of disease in a population. It has been defined as 'the study of the distribution and determinant of health related states or events in specified population, and the application of this study to the control of health problems' (Last 1988).

It is well known that the spread of various infectious diseases in human populations, is one of the most serious problems that modern society faces today. It causes mortality of millions of people as well as expenditure of enormous amounts of money in health care and control of diseases. Thus it is essential that adequate attention must be paid to the study and the control of such diseases.

It is observed that the transmission of communicable diseases mainly depends upon: (i) the susceptible population, (ii) the infective population, (iii) the reservoir population, (iv) the mode of transmission. In the following, we provide introductory information about transmission of infectious diseases as given by Park (1997).
Susceptibles are those who are prone to be infected and infectives are those who are already infected. The source or reservoir of infection is defined as 'the person, animal, arthropod plant, soil or substances (or any combination of these) in which an infectious

agent lives and multiplies, on which it depends primarily for survival, and where it reproduces itself in such a manner that it can be transmitted to a susceptible host' (Benenson 1981, Park 1997).

The reservoir may be of three types :

1. Human reservoir, 2. Animal reservoir, and 3. Reservoir in non-living things.

All communicable diseases are transmitted from the reservoir or source of infection to the susceptible host. They may be transmitted in many different ways, depending upon the infectious agent, portal of entry, environmental and ecological conditions. The mode of transmission of infectious disease may be direct or indirect as described below in Table 1 and Table 2 (Last 1988, Benenson 1981).

Table 1. **Direct Transmission**

Type	Pathway for transmission	Diseases
Direct contact	Direct contact from skin to skin, mucosa to mucosa or mucosa to skin of the same or another person, sexual contacts	STD and AIDS, Leprosy, Skin and Eye Infection
Droplet infection	Direct projection of a spray of droplets of saliva and naso-pharyngeal secretions during coughing, sneezing, spitting or talking into the surrounding atmosphere	Respiratory Infection Eruptive Fevers, Common Cold, Many Infections of the Nervous System, Diphtheria, Whooping Cough, Tuberculosis, Meningococcal, Meningitis, etc.
Contact with soil	Direct exposure of susceptible tissue to the disease agent in soil, compost or decaying vegetable matter	Hookworm Larvae, Tetanus, Mycosis, etc.
Inoculation into skin or mucosa	Disease agent may be inoculated directly into the skin or mucosa	Rabies, Hepatitis B.
Transplacental or vertical transmission	Disease agents can be transmitted transplacentally from mother to child	AIDS, Syphilis, Rubella, Measles, etc.

Table 2. **Indirect Transmission**

Type	Pathway for transmission	Diseases
Vehicle-borne	Transmission of the infectious agent through the agency of water, food, ice, blood, serum, plasma or other biological products such as tissues and organs	Acute Diarrheas, Typhoid Fever, Cholera, Polio, Hepatitis A, Food Poisoning Intestinal Parasites, etc.
Vector-borne	Vectors transmit infection by inoculation into the skin or mucosa by biting or by deposit of infective material on the skin or on the food or other objects	Diarrhea, Dysentery, Typhoid Fever by House Fly, Malaria
Air-borne transmission		
(a) Droplet nuclei	Small droplets evaporate rapidly leaving behind a minute residue or nucleus which may be a virus particle or bacteria and being very light they may remain air-borne for considerable periods	Influenza, Chicken-pox, Measles, Tuberculosis Typhoid Fever, etc.
(b) Infected Dust	When a person coughs or sneezes, larger drops of moisture settle on the floor and become part of the dust, so when the dust is inhaled, the person acquires infection	Tuberculosis, Pneumonia, etc.
Fomite-borne	Soiled clothes,towels, door handles, chains, syringes, instruments and surgical dressings	Diphtheria, Typhoid Fever, Bacillary Dysentery, Hepatitis A, Eye and Skin Infection
Unclean hands and fingers	Hands are the most common medium by which pathogenic agents are transferred to food from the skin, nose, towels, etc. as well as from other foods	Streptococcal Infections, Typhoid Fever, Dysentery, Hepatitis

1.1.1 Stages for Parasitism in a Susceptible Host

There are four stages in successful parasitism (Last 1988) :

(a) The infectious agent must find a portals of entry by which it may enter the host. There are many portals of entry, e.g. the respiratory tract, the alimentary tract, the genito-urinary tract, skin etc. Some organisms may have more than one portal of entry, e.g. hepatitis B, Q. fever.

(b) On gaining entry into the host, the organism must reach the appropriate tissue in the body of the host where it may find optimum conditions for its multiplication and survival.

(c) The disease agent must find a way out of the body (portal of exit) so that it may reach a new host and propagate its species. If there is no portal of exit, the infection becomes a dead end infection as in rabies, bubonic plague, tetanus, etc.

(d) After leaving the human body, the organism must survive in the external environment for a sufficient period until a new host is found. In addition, a successful disease agent should not cause the death of the host but produce only a low-grade immunity so that the host is vulnerable again and again to the same infection. The best example is common cold virus.

1.1.2 Effects of Demographic, Environmental and Ecological Changes on the Spread of Diseases

One of the most important factors responsible for the spread of diseases is the demographic changes that have occurred in our society on a global basis. Population growth in the past 50 years, since the end of World War II, has been phenomenal, especially in tropical developing countries. Cities have just expanded in an unplanned fashion, causing a deterioration in the housing, water, sewage and waste management systems. This creates ideal conditions for the spread of various infectious diseases (e.g. water borne, mosquito-borne, rodent-borne diseases) in these urban areas. Further household discharges dumped in open space on the road, particularly in areas of the city where poor

people live also contribute to the growth of carriers, bacteria and vectors, causing the spread of infectious diseases. These factors have contributed to the resurgence of infectious diseases in general, but vector-borne diseases in particular. Changes in agricultural practices have also influenced the resurgence or the emergence of vector-borne diseases. Building new dams which flood areas gives more area for mosquitoes to breed. Some irrigation schemes increase mosquito breeding and their population. Clearing forests and moving into previously unoccupied areas put people in greater contact with potential carriers, bacteria and vectors for infectious diseases.

1.1.3 Effect of Societal Changes on the Spread of Diseases

There are lots of societal changes as well as modern living conditions that cause the spread of infectious diseases, e.g. AIDS, dengue fever, malaria and so on.
(i) Plastics: Many of our consumer goods are now packaged in non-biodegradable plastics and cellophane. Because of mismanagement, several of these get into the environment, collect water, and make ideal larval habitats for all kinds of mosquitoes.
(ii) Automobiles: There has been an explosion in the automobile industries in the last several decades. The tires of these automobiles are discarded after use. They are non-biodegradable, and are not only good breeding grounds for mosquitoes, but also they are rat harborages.
(iii) Commerce: Shipping containers provide a vehicle for moving mosquito vectors and vermin around the world.
(iv) Air Travel: It provides the ideal mechanism to constantly move pathogens between population centers, especially urban diseases such as dengue, from one part of the country or the World to the other.
(v) Migration: Habitat to habitat migration, due to social, economic, ecological, environmental or political causes, brings infectives in contact with susceptibles in the new habitat and thus spreading the disease.

It is therefore essential to model and analyze the effects of various factors on the spread

of infectious diseases and predict the consequences.

In the following we give a brief literature survey of mathematical modelling of the spread of infectious diseases so that research work on modelling and analysis of the problems presented in the thesis can be seen in its proper perspective.

1.2 Brief Literature Survey on Modelling of Infectious Diseases

The mathematical theory of infectious diseases, pioneered by Ross, Kermack and McKendric, has been an important applied tool, especially for the establishment of vaccination strategies. There have been many notable contributions involving modelling of specific infectious diseases such as influenza (Liu and Levin 1989, Castillo-Chavez et al. 1988, 1989), rubella (Hethcote 1989), Japanese Encephalitis (Tapaswi et al. 1995) and AIDS (Anderson and May 1987, Castillo-Chavez et al. 1990, Castillo-Chavez 1989), (see also Hethcote 1976, Hethcote and Yorke 1984, Bailey 1957, 1975, 1979, 1980, 1982, Anderson and May, 1979 and May and Anderson 1979).

In general the spread of such diseases in human populations depends upon various factors such as the numbers of infectives, susceptibles, modes of transmission (carriers, vectors etc.), social, cultural and economic factors, environmental, ecological and geographical conditions (Fuzzi et al. 1996, Dufer 1982). An account of the modelling and study of epidemic diseases can be found in lecture notes by Waltman (1974), Hethcote (1974) and in the monographs by Bailey (1975, 1982) and the book by Levin et al. (1989).

As pointed out earlier many infectious diseases are spread by direct contact between susceptibles and infectives, while others get transmitted indirectly, for example tuberculosis, typhoid, cholera, malaria, etc. Some diseases spread in the environment and are transmitted to the human population by carriers, insects or vectors, which grow in the environment due to various household discharges (Cooke 1979, Marcati and Pozio

1980, Cairncross and Feachem 1983). The diseases such as tuberculosis, typhoid etc. are transmitted both ways i.e. directly as well as indirectly. In direct transmission susceptibles get infected through meeting or mixing with infectives, while in the case of indirect transmission bacteria enter the environment and then contaminate food or water which may be later consumed by susceptibles (Hethcote 1976, Gonzalez-Guzman 1989, Shukla 1986).

The asymptotic behavior and stability of non-linear epidemic models have been investigated by many researchers Bailey (1979, 1980, 1982), Cooke (1979), Dietz (1979, 1982), Hethcote (1973, 1974, 1976, 1981), Hethcote et al (1973), Kamper (1978) and Wichmann (1979). In particular, Hethcote (1976) presented a qualitative analysis of several non-linear models with vital dynamics and with carriers of constant density. Greenhalgh (1990) considered SIS models with density dependent death rate and Greenhalgh and Das (1992) extended this model for a variable contact rate. In most models quoted above the latent period has been assumed negligible but there are diseases in which the latent period is important. This aspect can be considered by including an exposed class in the usual epidemic models (Hethcote and Driessche 1991, Hethcote 1994). Trawis and Lenhart (1987) and Sexena et al. (1993) have analyzed an SIR epidemic model with a heterogenous population with the assumption that individuals of different communities have different rates of contact. Some models related to nonlinear contact rates (dependent upon susceptibles, infectives or population size) have also been proposed and analyzed (Capasso and Serio 1978, Hethcote 1994, Liu et al. 1986, 1987). In most of the models of infectious diseases it is assumed that the total population size remains constant as birth and death rates are equal. But this does not happen if the death rate caused by disease is significantly large as in the case of cholera. A similar situation also arises if there is no balance between incoming and outgoing population from the region under consideration (Hethcote 1994). In view of this, models with demographic structure have been proposed and analyzed by considering variation of total population size, which involves birth rate, death rate, immigration etc. (Brauer 1995, Hethcote 1994, Mena-Lorca and Hethcote 1992, Zhou and Hethcote 1994). In particular Hethcote and Van den Driessche (1995)

analyzed an SIS model with variable population size and a delay. Gao and Hethcote (1992) and Gao et al. (1995) discussed some epidemic models with density dependent birth and death rates and also models with periodicity. Zhou and Hethcote (1994) analyzed epidemic models with population size dependent incidence for diseases without immunity.

In the papers mentioned above, indirect transmission of diseases is not considered. Hethcote (1976) analyzed an SIR model with constant density of carriers. Gonzalez-Guzman (1989) discussed an SIS model for typhoid fever by considering the effect of flow of bacteria from infected population in the environment through sewage which then affects the susceptible population by contaminating drinking water.

Modelling of the transmission of malaria started in the early part of the nineteenth century (Ross 1911, 1929, Lotka 1923). Ross (1911, 1921) constructed an epidemiological model to show that if the mosquito population density is reduced below a threshold level, the rate of getting new infections would fall below the rate at which infected persons recover leading to elimination of malaria. Since then, various investigators have studied the spread of malaria using both deterministic and stochastic models (Bailey 1979, 1982, Dietz et al. 1974, Macdonald 1953, 1957, Manoharan et al. 1996, Molineaux et. al. 1978 and Radcliffe 1973, 1974). In particular Radcliffe (1973, 1974) has studied the utility of catalytic models in the estimation of incidence and prevalence of malaria in a hyper-endemic situation. It has been suggested that the models for malaria involve two populations of host (human) and vector (mosquito), and are similar in mathematical structure to gonorrhea epidemic models (Bailey 1979, 1982, Hethcote and Yorke 1984, Lajmanovich and Yorke 1976, Nallaswamy and Shukla 1982).

It may also be noted here that spatial migration of population also plays a very important role in the spread of infectious diseases. In countries like India having large population density and population distributed over a large number of villages with small inter-village travelling distances, deterministic models for infectious diseases involving a spatial aspect may be quite suitable. Various attempts have been made to study geographical spread

of infectious diseases by considering dispersive migration of susceptibles and infectives (Capasso 1978, Capasso and Maddalena 1981, Marcati and Pozio 1980, Webb 1981). In particular, Marcati and Pozio (1980) have investigated the global behaviour of a vector disease model by considering spatial spread and hereditary effects which may be applicable to growth and spread of malaria. A theoretical expression giving the velocity of propagation for geographical spread of host-vector and carrier borne epidemics has been developed by Radcliffe (1973). The effects of cross dispersal, which arise due to spatial influence of one species on another species, have also been considered in the study of infectious diseases (Bailey 1980 and Radcliffe 1973). More details regarding spatial spread of epidemic diseases, both deterministic and stochastic, can be found in the review papers by Bailey (1980) and Mollison (1977, 1995). Effects of dispersal on the linear and non-linear stability of the epidemic equilibrium state of the system governing the spread of gonorrhea have been investigated by Cooke and Yorke (1973) and Nallaswamy and Shukla (1982).

1.2.1 Environmental and Demographic Effects on Modelling of Spread of Infectious Diseases

From the above survey it can be noted that little effort has been made to model the effect of environmental changes on the spread of infectious diseases transmitted by carriers, bacteria or vectors, whose population may be dependent on such changes. For example, household discharges may be very conducive to the growth of carrier populations, such as flies, cockroaches as well as bacterial populations and vector populations such as mosquitoes. This increase helps in enhancing the contact rate of infections with susceptibles leading to fast spread of the disease (Shukla 1986, Shukla et al. 1987, Misra 1987).

It is noted here that the spread of an infectious disease is governed by factors such as: (i) the density of the human population, (ii) the density of the carrier, bacteria or vector population, (iii) environmental factors such as household discharges, rain, tem-

perature humidity etc, (iv) ecological factors such as vegetation, biomass density and (v) geographical factors. But the study of the spread of an infectious disease using mathematical models by considering these factors has not been done, particularly when the density of the human and carrier, bacteria or vector populations are variables. By considering the example of malaria, it may be noted that in highly endemic areas, such as in parts of Africa or the north-eastern part of India, persons who have been repeatedly infected with parasites of malaria acquire a degree of immunity which suppresses most clinical symptoms. These people may carry gametocytes in their blood that infect the mosquitoes biting them and form a separate class of reservoir population which helps in spreading malaria without being affected themselves. The effect of the reservoir population on the spread of malaria or any other infectious disease has also not been studied using mathematical models. Both the effects are important and need to be modeled and analyzed.

A common phenomenon is the migration of population within the same region due to economic, social, religious, political, environmental or other considerations. The migrating population carries with it all its traditional values, cultural heritage and so on, including diseases if any are present in the population. After movement of the population into the new habitat the susceptibles join the new susceptible class and the infectives join the new infective class and the usual interaction begins. The same phenomenon exists between two socially structured populations (rich and poor) living in two environmentally different regions, one is cleaner and the other is affected by various discharges caused by the population in that region and a fraction of this population moves to the environmentally better region. Further, it may happen that the poor population living in the environmentally degraded habitat spread the disease by interacting with the rich population by working in their habitat as service providers (without migration). All these phenomena need to be modeled and analyzed.

In the modelling of infectious diseases, as suggested above, one can use simple mass action incidence for interaction of susceptibles with infectives or the true mass action incidence.

We have been guided by the fact that in most tropical third world countries, where population size changes due to immigration, births and deaths, the population density does not remain constant and generally the contact rate is proportional to the size of susceptibles and infectives (Anderson and May 1979, de Jonge et al. 1995). To make this case very specific, one may consider labor colonies in cities like Bombay, Calcutta, Kanpur, etc. In these colonies, the area of which may be several square kilometers, working class people live with very little municipal facilities such as clean water, bathrooms, toilets, etc. Further these people due to lack of civic facilities, throw household wastes in the open space on or around the road. A fraction of the garbage in these dumps becomes the breeding ground for carriers, vectors or some harmful bacteria. This unhygenic situation enhances the growth of carriers, vectors and bacteria, causing the fast spread of infectious disease. The population density per square kilometer of such colonies can vary from about 50,000 to 1,50,000. In joint families, grown up youngsters get married and children continue to live with their parents, thus increasing the population density. Further these colonies also work as immigration centers. Laborers from the countryside migrate to these colonies and rarely return home. This further increases the population density of these colonies. Hence we feel in such situations simple mass action incidence may be more appropriate while modelling the spread of infectious diseases, where rate of contact may be directly proportional to population density (Mena-Lorca and Hethcote 1992).

To be specific, the following problems have been considered selectively to provide samples of modelling of environmental and demographic effects on the spread of infectious diseases:

(i) Modelling the Spread of Carrier-Dependent Infectious Diseases with Environmental and Demographic Effects,

(ii) Modelling Bacterial Disease with Environmental and Demographic Effects,

(iii) Modelling the Spread of Malaria: Environmental and Demographic Effects,

(iv) Modelling the Spread of Malaria with Human Reservoir: Environmental and Demographic Effects,

(v) Modelling the Spread of a Carrier-Dependent Infectious Disease in Two Neighboring Habitats with Migration in Between,
(vi) Modelling the Spread of Bacterial Disease in a Population: Effect of Service Providers from a Environmentally Degraded Region.
The density of the carrier, bacteria or vector population has been assumed to follow a logistic growth model, the growth rate of which further enhances as the cumulative rate of household discharges increases. These problems have been modelled and analyzed by using the variational matrix method, Liapunov's second method and computer simulation (La Salle and Lefschetz 1961).

1.3 Summary of the thesis

The thesis consists of seven chapters and is divided into two parts. The first part deals with modelling infectious diseases in a single population whereas the second part deals with the spread of infectious diseases into two populations living in two different environmental conditions.

The first chapter provides the introduction regarding the dynamics of infectious diseases including a brief literature survey relevant to modelling and analysis of infectious diseases, so that the work done in the thesis can be seen in its proper perspective.

In the second chapter, we study the effects of the following factors on the spread of a carrier-dependent infectious disease using SIS mathematical models:
(i) A variable carrier population, the density of which follows logistic model and increases further due to household discharges in the habitat,
(ii) human demography.
The following two types of demographic factors are considered in the analysis:
(i) a population with constant immigration and death rate,
(ii) a population with logistic growth.
It is shown that the endemic equilibrium is globally stable when the rate of cumulative

environmental discharges, conducive to the growth of carrier population, is a constant. When the rate is a function of human population density, the endemic equilibrium is also locally and globally stable but under certain conditions. The global stability is also illustrated by computer simulation. It is shown that if the human population increases due to demographic changes, then the spread of the infectious disease increases. It is further noted from the analysis that the spread of the infectious disease increases as the growth rate of carrier population (caused by conducive environmental discharges due to human sources) increases.

In the third chapter, we have considered a bacterial disease model, which is very much similar to the models of Chapter 2 involving carriers. Here the only difference is that bacteria are also released by infectives into the habitats continuously and their growth rate increases not only because of household discharges but also by continuous release from infectives. Here the SIS models for bacterial infectious diseases, (like tuberculosis, typhoid, etc.), which are caused by direct contacts of susceptibles with infectives and also by bacteria, are proposed and analyzed quantitatively as well as by simulation. It is found again that the endemic equilibria are globally stable when the rate of cumulative environmental discharges is constant. But when the rate of discharge is a function of human population density, the endemic equilibrium is locally and globally stable under certain conditions only. It is further shown that under various demographic situations mentioned above, the infective population increases with the increase of household discharges conducive to the growth of bacteria population and the spread of the disease increases and so it becomes more endemic.

In Chapters 4 and 5, malaria models without and with reservoir have been proposed and analyzed by considering,
(i) the effect of cumulative discharges from household sources and (ii) the effects of demographics of human and mosquito populations.
The case of constant as well as variable discharge rates are considered. The threshold parameter for spread of the diseases is derived in each case. If this threshold is greater

than one, the nontrivial equilibrium in the corresponding case is always feasible and is locally asymptotically stable to small perturbations. It is also shown by analysis and simulation that this equilibrium is globally stable under certain conditions. The effect of demographic increases in human population is found to increase the spread of malaria. It is observed that as the cumulative environmental discharges increase, the mosquito population increases, leading to fast spread of Malaria.

In Chapter 6, a model is considered to study the effect of immigration of population from an environmentally degraded habitat to an environmentally cleaner habitat on the spread of an infectious disease. The constant as well as the variable rates of household discharges are considered. In each case, the conditions for existence of endemic equilibrium points have been obtained. It has been shown that under certain conditions, these equilibria are locally stable showing that the disease is endemic. Further by simulation, it is observed that the endemic equilibrium points are in fact globally stable under local stability conditions. It is concluded that as the migration rate increases, the total and the infective population densities of the region with cleaner environmental conditions increase, whereas the total and the infective population densities of the region of degraded environmental conditions decrease.

Lastly in Chapter 7, a mathematical model is proposed by keeping in view that a poor population living in an environmentally degraded habitat spreads the disease by interacting with the rich population by working in their habitat as service providers (without migration). The model is presented under similar environmental conditions as in previous chapters, here also linear and nonlinear analysis of the equilibria have been studied for various cases. It is noticed that as the interaction between rich and poor class increases, the disease spreads more rapidly in the rich class as expected. It is suggested that it is in the interest of the rich class to not only clean their own environment but also to clean the environment of the habitat, in their neighborhood, where poor people live.

Chapter 2

Modelling the Spread of Carrier-Dependent Infectious Diseases with Environmental and Demographic Effects

2.1 Introduction

Many infectious diseases spread by carriers such as flies, ticks, mites and snails, which are present in the environment (Harold 1960, Harry and John 1962, Harry and Kent 1961). For example, air-borne carriers or bacteria spread diseases such as tuberculosis and measles; while water-borne carriers or bacteria are responsible for the spread of dysentery, gastroenteritis, diarrhea, etc. (Cairncross and Feachem 1983, Taylor and Knowelden, 1964). Various kinds of household and other wastes, discharged into the environment in residential areas of population, provide a very conducive environment for the population growth of some of these carriers (Ludwig 1975, Purdom 1980). This enhances the chance of carrying more bacteria from infectives to the susceptibles in the population leading to fast spread of carrier dependent infectious diseases. Thus unhygienic environmental conditions in the habitat caused by humans population become responsible for the fast spread of an infectious disease.

In recent decades, there have been several investigations of infectious diseases using deter-

ministic mathematical models with or without demographic change (Bailey 1979, 1980, Gao and Hethcote 1992, Gonzalez-Guzman 1989, Greenhalgh 1990, 1992, Greenhalgh and Das 1995, Hethcote 1976, 1994, Mena-Lorca and Hethcote 1992, Zhou and Hethcote 1994). In particular Greenhalgh (1992) has studied an infectious disease model with population dependent death rate using computer simulation. Gao and Hethcote (1992) analyzed an infectious disease model with logistic population growth. Zhou and Hethcote (1994) have studied a few models for infectious diseases using various kinds of demographics. Hethcote (1976) has discussed an epidemic model in which the carrier population is assumed to be constant. But in general the size of the carrier population varies and depends on the natural conditions of the environment as well as on various discharges into it by the human population.

Thus in this chapter, the effects of the following factors on the spread of an infectious disease governed by SIS mathematical models are studied:

(i) a variable carrier population caused by environmental discharges in the habitat,

(ii) human demography.

The following two types of demographic factors are considered in the analysis:

(i) a population with constant immigration, and

(ii) population with logistic growth.

2.2 SIS Model with Constant Immigration

In this chapter an SIS model with immigration is considered, where the population density N(t) is divided into two classes: susceptibles X(t) and infectives Y(t). It is assumed that all susceptibles living in the habitat are affected by a carrier population of density C(t), which grows logistically with given intrinsic growth rate and carrying capacity. The growth rate of its density is further assumed to increase with the increase in the cumulative density of discharges by the human population into the environment. Keeping the above in mind and by considering simple mass action interaction, a mathematical

model is proposed as follows,

$$\begin{aligned}
\dot{X} &= A - dX - \beta\, X\, Y - \lambda\, X\, C + \nu\, Y, \\
\dot{Y} &= \beta\, X\, Y + \lambda\, X\, C - (\nu + \alpha + d)\, Y, \\
\dot{N} &= A - dN - \alpha\, Y, \\
\dot{C} &= s\, C \left(1 - \frac{C}{L}\right) - \delta\, C + s_1\, E\, C, \\
\dot{E} &= Q(N) - \delta_0\, E, \\
X &+ Y = N,
\end{aligned} \tag{2.1}$$

$X(0) = X_0 > 0,\ Y(0) = Y_0 \geq 0,\ N(0) = N_0 > 0,\ C(0) = C_0 \geq 0$, and $E(0) = E_0 > 0$.

Here $E(t)$ is the cumulative density of environmental discharges conducive to the growth of carrier population; A is the constant immigration rate of the human population; d is the natural death rate constant; β and λ are the transmission coefficients due to the infectives and the carrier population respectively; α is the disease related death rate constant and ν is the recovery rate constant i.e the rate at which individual recovers and moves to the susceptible class again from the infective class. The constant L is the carrying capacity of the carrier population in the natural environment; s is its intrinsic growth rate; δ is the death rate of carriers due to control measures, where $s > \delta$; s_1 is the per capita growth rate coefficient of the carrier population due to the cumulative environmental discharges rate Q(N) which is human population density dependent (an increasing function of N) and δ_0 is the depletion rate coefficient of the environmental discharges. In writing the model (2.1), we use the term transmission coefficient in the sense as used by Anderson and May 1983 (see also de Jong et al., 1995), which means that new cases of disease occur at the rates βXY and λXC due to interaction of susceptibles with infectives and carriers respectively.

It can be seen that the region of attraction

$$T = \left\{ (Y, N, C, E) :\ 0 \leq Y \leq N \leq \frac{A}{d},\ 0 \leq C \leq \frac{L}{s}\left(s - \delta + s_1\, \frac{Q(\frac{A}{d})}{\delta_0}\right),\ 0 \leq E \leq \frac{Q(\frac{A}{d})}{\delta_0} \right\},$$

is positively invariant and all solutions starting in T stay in T. The continuity of the right hand sides of (2.1) and their derivatives imply that a unique solution exists (Hale

1969).

The model (2.1) is analyzed in the following two cases:

(i) the rate of cumulative environmental discharges Q is a constant,

and

(ii) the rate of cumulative environmental discharges Q is a function of the population density. The function $Q(N)$ is such that it satisfies following conditions:

$$Q(0) = Q_0 > 0, \quad Q'(N) \geq 0,$$

i.e Q is an increasing function of N. We consider the form of $Q(N)$ as $Q(N) = Q_0 + lN$, where $l > 0$ is a constant.

2.2.1 Case I: $Q = Q_a$, a Constant

To analyze the model (2.1) in this case, we consider the following equivalent system, since $X + Y = N$.

$$\begin{aligned}
\dot{Y} &= \beta\,(N-Y)\,Y + \lambda\,(N-Y)\,C - (\nu+\alpha+d)\,Y, \\
\dot{N} &= A - dN - \alpha\,Y, \\
\dot{C} &= s\,C\left(1-\frac{C}{L}\right) - \delta\,C + s_1\,E\,C, \\
\dot{E} &= Q_a - \delta_0\,E.
\end{aligned} \tag{2.2}$$

From the last two equations of (2.2), it is seen that

$$\limsup_{t\to\infty} E(t) = \frac{Q_a}{\delta_0} \text{ and } \limsup_{t\to\infty} C(t) = \frac{L}{s}\left(s-\delta+s_1\,\frac{Q_a}{\delta_0}\right) = C_m > 0.$$

Thus to see the global behavior of the system it is reasonable to reformulate system (2.2) as follows:

$$\begin{aligned}
\dot{Y} &= \beta(N-Y)\,Y + \lambda(N-Y)\,C_m - (\nu+\alpha+d)\,Y, \\
\dot{N} &= A - d\,N - \alpha\,Y.
\end{aligned} \tag{2.3}$$

Remark: It may be noted that C_m increases as s_1, Q_a and L increase or as δ and δ_0 decrease.

The result of equilibrium analysis is stated in the following theorem.

THEOREM 2.1 There exists a nontrivial equilibrium $P^*(\hat{Y}, \hat{N})$ regulating the population density, where

$$\hat{Y} = \frac{H + \sqrt{H^2 + 4\beta\,(1 + \frac{\alpha}{d})\,\lambda\,\frac{C_m\,A}{d}}}{2\beta\,\left(1 + \frac{\alpha}{d}\right)}, \quad \hat{N} = \frac{A - \alpha\hat{Y}}{d}$$

$$\text{and} \quad H = \left\{\frac{\beta\,A}{d} - \lambda\,(1 + \frac{\alpha}{d})\,C_m - (\nu + \alpha + d)\right\}.$$

Proof: Setting the right hand side of (2.3) to zero, it is noted that $\hat{Y}$ is the positive root of the following quadratic:

$$F(Y) \;=\; \beta\left(1 + \frac{\alpha}{d}\right)\,Y^2 - \left\{\beta\,\frac{A}{d} - \lambda\,C_m\left(1 + \frac{\alpha}{d}\right) - (\nu + \alpha + d)\right\}\,Y - \lambda\,C_m\,\frac{A}{d} = 0 \tag{2.4}$$

and $\hat{N} = \frac{A - \alpha\hat{Y}}{d}$. As $F(0) \;=\; -\lambda C_m \frac{A}{d} \;<\; 0$ and $F(\frac{A}{\alpha+d}) \;>\; 0$, $\hat{Y}$ lies between 0 and $\frac{A}{\alpha+d}$, which implies $\hat{N}$ is positive.

Remark: From (2.4), it can be seen that $\frac{d\hat{Y}}{dC_m} \;>\; 0$. This implies $\hat{Y}$ increases as C_m increases or as any of Q_a, L or s_1 increases. It may be pointed out that increase in Q_a gives a more conducive environment to carrier population growth and so spread of the epidemic increases.

2.2.1.1 Stability Analysis

The stability result of the equilibrium P^* is stated in the following theorem.

THEOREM 2.2 The equilibrium P^* is locally as well as globally asymptotically stable.

Proof: Let $\hat{M}$ be the variational matrix corresponding to the system (2.3) at the equilibrium point $(\hat{Y}, \hat{N})$, then

$$\hat{M} = \begin{pmatrix} -\left(\beta\hat{Y} + \dfrac{\lambda\,\hat{N}\,C_m}{\hat{Y}}\right) & \beta\,\hat{Y} + \lambda\,C_m \\ -\alpha & -d \end{pmatrix}.$$

From the above variational matrix, it is noted that equilibrium point $P^*(\hat{Y}, \hat{N})$ is locally asymptotically stable.

It can also be proved that, this equilibrium point is globally asymptotically stable by using the following Liapunov function about $P^*(\hat{Y}, \hat{N})$ as follows;

$$V = \left(Y - \hat{Y} - \hat{Y} ln\frac{Y}{\hat{Y}}\right) + \frac{1}{2}\, k_1\, (N - \hat{N})^2, \tag{2.5}$$

where k_1 is to be suitably chosen. From (2.3) and (2.5), we get

$$\begin{aligned}
\dot{V} &= (Y-\hat{Y})[\beta(N-\hat{N}) - \beta(Y-\hat{Y}) + \lambda C_m \left(\frac{N}{Y} - \frac{\hat{N}}{\hat{Y}}\right)] \\
&\quad + k_1(N-\hat{N})[-d(N-\hat{N}) - \alpha(Y-\hat{Y})] \\
&= -\left(\beta + \frac{\lambda C_m N}{Y\hat{Y}}\right)(Y-\hat{Y})^2 - k_1 d(N-\hat{N})^2 + \left\{\beta + \frac{\lambda C_m}{\hat{Y}} - k_1\alpha\right\}(N-\hat{N})(Y-\hat{Y}).
\end{aligned}$$

Choosing $k_1 = \frac{\beta\hat{Y} + \lambda C_m}{\alpha\hat{Y}}$, it is seen that $\dot{V}$ is negative definite, implying global stability of this equilibrium point P^*. The above theorem implies that the spread of a carrier-dependent infectious disease increases as the carrier population density increases due to household discharges.

2.2.2 Case II: $Q = Q_0 + lN$

In this case, using $X + Y = N$, the model (2.1) is rewritten in the following form:

$$\begin{aligned}
\dot{Y} &= \beta\,(N-Y)\,Y + \lambda\,(N-Y)\,C - (\nu+\alpha+d)\,Y, \\
\dot{N} &= A - dN - \alpha\,Y, \\
\dot{C} &= s\,C(1-\frac{C}{L}) - \delta\,C + s_1\,E\,C, \\
\dot{E} &= Q(N) - \delta_0\,E = Q_0 + lN - \delta_0 E,
\end{aligned} \tag{2.6}$$

$$X(0) = X_0 > 0,\ Y(0) = Y_0 \geq 0,\ N(0) = N_0 > 0,\ C(0) = C_0 \geq 0 \text{ and } E(0) = E_0 > 0.$$

Now we discuss the equilibrium analysis of the above system and results of the analysis are summarized in the following theorem.

THEOREM 2.3 There exist the following three equilibria, namely

(i) $P_1\ \left(0,\ \frac{A}{d},\ 0,\ \frac{Q_0 + l\frac{A}{d}}{\delta_0}\right)$,

(ii) $P_2\ \left(\bar{Y},\ \bar{N},\ 0,\ \frac{Q_0 + l\bar{N}}{\delta_0}\right)$, *which exists if* $\beta\frac{A}{d} - (\nu+\alpha+d) > 0$,

where $\bar{N} = \dfrac{\beta\frac{A}{d} + (\nu+\alpha+d)\frac{\alpha}{d}}{\beta(1+\frac{\alpha}{d})}$ *and* $\bar{Y} = \dfrac{\beta\frac{A}{d} - (\nu+\alpha+d)}{\beta(1+\frac{\alpha}{d})} > 0$,
and (iii) P_3 $(\hat{Y},\ \hat{N},\ \hat{C},\ \hat{E})$.

Proof: The existence of equilibrium P_1 or P_2 is obvious. We prove the existence of P_3 as follows. The equilibrium point $P_3(\hat{Y}, \hat{N}, \hat{C}, \hat{E})$ is given as the solution of the following system of equations,

$$E = \frac{Q_0 + lN}{\delta_0}, \quad C = \frac{L}{s}\left\{s - \delta + s_1\,\frac{Q_0 + lN}{\delta_0}\right\}, \tag{2.7}$$

$$Y = \frac{A}{\alpha} - \frac{d}{\alpha}\,N, \tag{2.8}$$

and

$$\beta\,Y^2 - \beta N\,Y + (\nu+\alpha+d)Y + \lambda\frac{L}{s}\left\{s - \delta + s_1\frac{Q_0 + lN}{\delta_0}\right\}Y - \lambda\frac{L}{s}\{s - \delta + s_1\frac{Q_0 + lN}{\delta_0}\}N \ = \ 0. \tag{2.9}$$

Clearly, in the N-Y plane (2.8) is a straight line and (2.9) is a hyperbola, the positive branch of which lies in first, second and third quadrants.

Thus plotting the equations (2.8) and (2.9) in first quadrant (fig. 2.1), we get a branch of hyperbola passing through origin and intersecting the line (2.8), which gives $\hat{Y}$ and $\hat{N}$. Then $\hat{C}$ and $\hat{E}$ can be found by (2.7) and we get the third equilibrium point P_3.

Remark 1: From (2.9), the slope $\left(\frac{dY}{dN}\right)$ is given by

$$\left(\frac{dY}{dN}\right) = \frac{Y\{\beta Y + \lambda C + \frac{\lambda L s_1}{s l \delta_0}(N - Y)\}}{(\beta Y^2 + \lambda C N)} \ > \ 0, \text{ for } Y > 0,\ N > 0.$$

$$\text{Also } \left(\frac{dY}{dN}\right)_{(0,0)} = \frac{\frac{\lambda L}{s}\,\{s - \delta + s_1\,\frac{Q_0}{\delta_0}\}}{\nu + \alpha + d + \frac{\lambda L}{s}\,\{s - \delta + s_1\,\frac{Q_0}{\delta_0}\}} \ > \ 0.$$

It is noted that $\left(\frac{dY}{dN}\right)_{(0,0)}$ increases as Q_0 increases for $s - \delta + s_1\frac{Q_0}{\delta_0} \ > \ 0$.

Remark 2: When $\lambda = 0$, i.e. in the absence of carrier population for existence of an endemic equilibrium, i.e. for $\hat{Y}$ to be positive, we must have a threshold as $\frac{\frac{\beta A}{d}}{\nu+\alpha+d} \ > \ 1$ (from (2.9)), which is same as mentioned in Gao and Hethcote (1992).

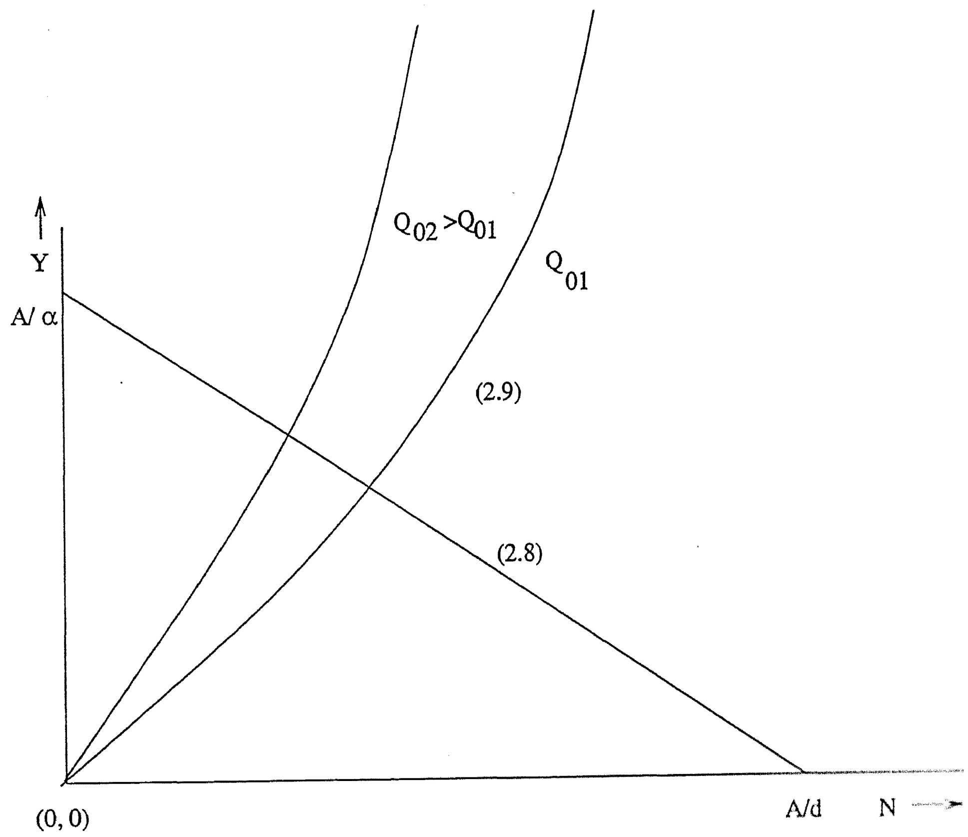


Figure 2.1: Existence of equilibrium point P_3 and effect of Q_0 on it.

2.2.2.1 Stability Analysis

Now we discuss the linear stability of equilibria P_1, P_2 and P_3 and nonlinear stability of the only non-trivial equilibrium P_3.
The local stability results of these equilibria are stated in the following theorem.

THEOREM 2.4 The equilibria P_1 and P_2 are locally unstable and the equilibrium P_3 is locally asymptotically stable provided

$$\begin{vmatrix} a_3 & a_1 & 0 \\ 1 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0,$$

where a_3, a_2, a_1 and a_0 are given in the proof of the theorem.

Proof: To prove the theorem, let M_i be the variational matrix corresponding to equilibrium points P_i, for $i = 1, 2, 3$, then

$$M_1 = \begin{pmatrix} \frac{\beta A}{d} - (\nu + \alpha + d) & 0 & \frac{\lambda A}{d} & 0 \\ -\alpha & -d & 0 & 0 \\ 0 & 0 & s - \delta + \frac{s_1}{\delta_0}(Q_0 + l\frac{A}{d}) & 0 \\ 0 & l & 0 & -\delta_0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} \beta \bar{N} - 2\beta\bar{Y} - \overline{\nu + \alpha + d} & \beta\bar{Y} & \lambda(\bar{N} - \bar{Y}) & 0 \\ -\alpha & -d & 0 & 0 \\ 0 & 0 & s - \delta + \frac{s_1}{\delta_0}(Q_0 + l\bar{N}) & 0 \\ 0 & l & 0 & -\delta_0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} -(\beta\hat{Y} + \frac{\lambda\hat{N}\hat{C}}{\hat{Y}}) & \beta\hat{Y} + \lambda\,\hat{C} & \lambda(\hat{N} - \hat{Y}) & 0 \\ -\alpha & -d & 0 & 0 \\ 0 & 0 & -\frac{s}{L}\,\hat{C} & s_1\,\hat{C} \\ 0 & l & 0 & -\delta_0 \end{pmatrix}.$$

From M_1 and M_2, it is clear that the equilibrium points P_1 and P_2 are locally unstable. To study stability of P_3, the characteristic polynomial corresponding to M_3 is obtained as

$$\psi^4 + a_3\,\psi^3 + a_2\,\psi^2 + a_1\psi + a_0 = 0,$$

where

$$\begin{aligned}
a_3 &= \beta\,\hat{Y} + \frac{\lambda\,\hat{N}\,\hat{C}}{\hat{Y}} + \frac{s}{L}\,\hat{C} + \delta_0 + d, \\
a_2 &= \left(\beta\,\hat{Y} + \frac{\lambda\,\hat{N}\,\hat{C}}{\hat{Y}}\right)\left(\frac{s}{L}\,\hat{C} + \delta_0 + d\right) + \frac{s}{L}\,\hat{C}\,\delta_0 + d\left(\frac{s}{L}\,\hat{C} + \delta_0\right) + \alpha(\beta\,\hat{Y} + \lambda\,\hat{C}), \\
a_1 &= \left(\beta\,\hat{Y} + \frac{\lambda\,\hat{N}\,\hat{C}}{\hat{Y}}\right)\left\{\frac{s}{L}\,\hat{C}\,\delta_0 + d\,(\frac{s}{L}\,\hat{C} + \delta_0)\right\} + \frac{s}{L}\hat{C}\,\delta_0\,d + \alpha(\beta\hat{Y} + \lambda\,\hat{C})\left(\frac{s}{L}\,\hat{C} + \delta_0\right), \\
a_0 &= \left(\beta\,\hat{Y} + \frac{\lambda\,\hat{N}\,\hat{C}}{\hat{Y}}\right)\frac{s}{L}\,\hat{C}\,\delta_0\,d + \alpha\lambda\,l\,s_1\hat{C}(\hat{N} - \hat{Y}).
\end{aligned}$$

Using the Routh-Hurwitz criteria, the conditions for local stability of this equilibrium are as follows:

$$a_3 > 0, \quad \begin{vmatrix} a_3 & a_1 \\ 1 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_3 & a_1 & 0 \\ 1 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0, \quad \begin{vmatrix} a_3 & a_1 & 0 & 0 \\ 1 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & 1 & a_2 & a_0 \end{vmatrix} > 0.$$

It is noted that first two inequalities are obvious and if the third inequality is satisfied so is the fourth one. Thus the equilibrium P_3 is locally asymptotically stable if only the third inequality is satisfied. Hence the theorem follows.

Nonlinear Analysis and Simulation

We first note that the system (2.6) is bounded by its corresponding system with $\alpha = 0$ and $Q(N)$ replaced by $Q(\frac{A}{d})$. Further, using comparison theorems (Lakshmikantham and Leela 1969), we observe that the non-trivial solution of the system (2.6) is bounded by its corresponding system with $\alpha = 0$ and $Q(N)$ replaced by $Q(\frac{A}{d})$, which can be shown to be globally stable following similar analysis as in Case I. Hence we speculate that the system (2.6) to be globally stable under the local stability condition in the interior of the region of attraction. To support this conjecture and to see the effects of various environmental and other parameters on the growth of the epidemic (i.e. on infective density) we integrate the system (2.6) by the fourth order Runge-Kutta method using the following set of parameters, which satisfy the local stability condition stated in the previous theorem.

$$\beta = 0.00000051, \quad \lambda = 0.000000021, \quad \nu = 0.012, \quad \alpha = 0.0005, \quad \delta = 0.6, \quad \delta_0 = 0.001,$$

$$A = 10, \quad d = 0.0004, \quad s = 0.9, \quad Q_0 = 20, \quad s_1 = 0.000002, \quad l = 0.00005, \quad L = 100000.$$

We may note here that all the parameters are in per day except the carrying capacity L which is just a number. The equilibrium values of $\hat{Y}$, $\hat{N}$, $\hat{C}$ and $\hat{E}$ have been found as $\hat{Y} = 3407.490$, $\hat{N} = 20740.596$, $\hat{C} = 38008.224$ and $\hat{E} = 21037.027$.

The computer simulation is performed for different initial positions in the following four cases,

1. $Y(0) = 100$, $N(0) = 12000$, $C(0) = 5003$ and $E(0) = 300$,
2. $Y(0) = 1800$, $N(0) = 24000$, $C(0) = 11000$ and $E(0) = 9000$,
3. $Y(0) = 8330$, $N(0) = 23000$, $C(0) = 4000$ and $E(0) = 100$,
4. $Y(0) = 4000$, $N(0) = 15500$, $C(0) = 9500$ and $E(0) = 455$.

In Fig. 2.2, the infective population is plotted against the susceptible population.

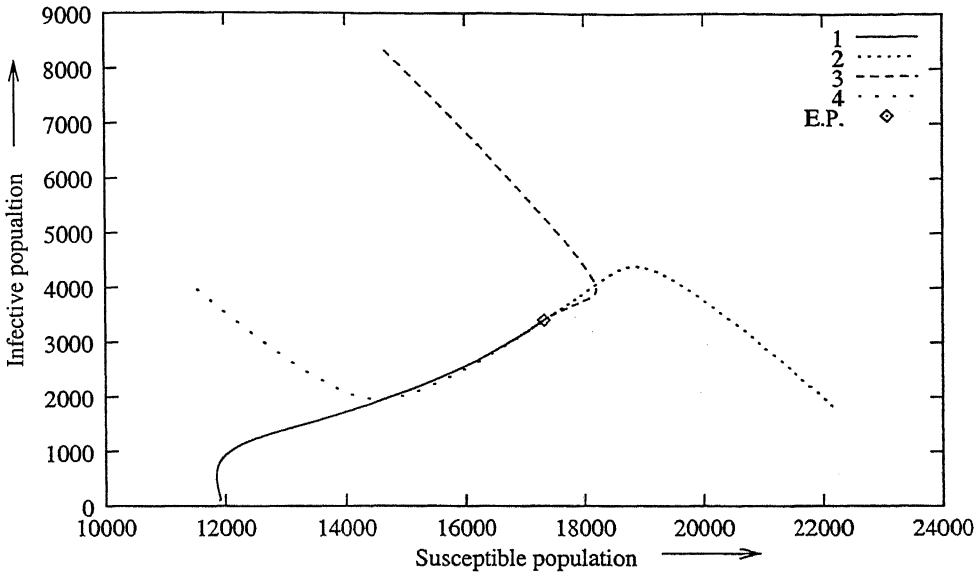


Figure 2.2: Variation of infective population with susceptible population.

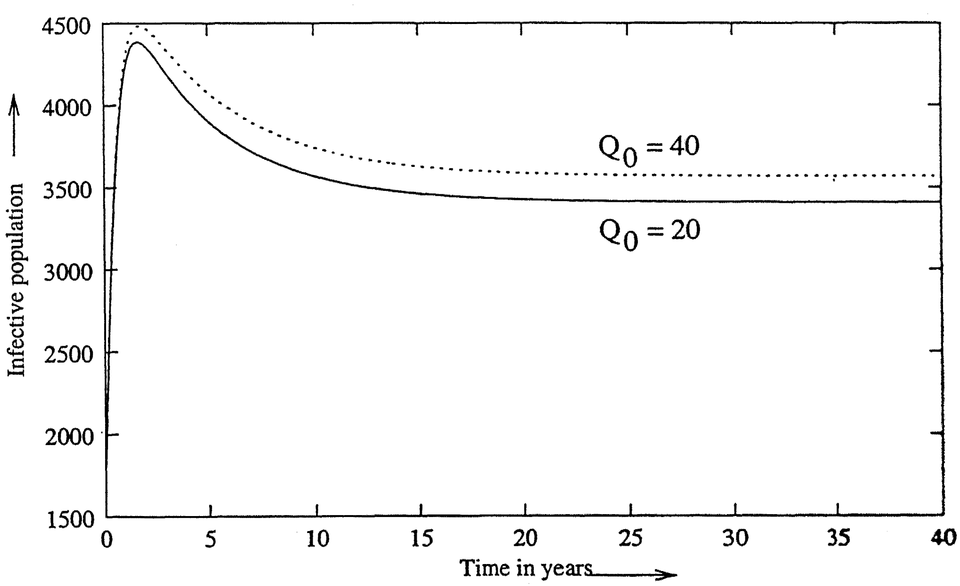


Figure 2.3: Variation of infective population with time for different cumulative environmental discharge rates.

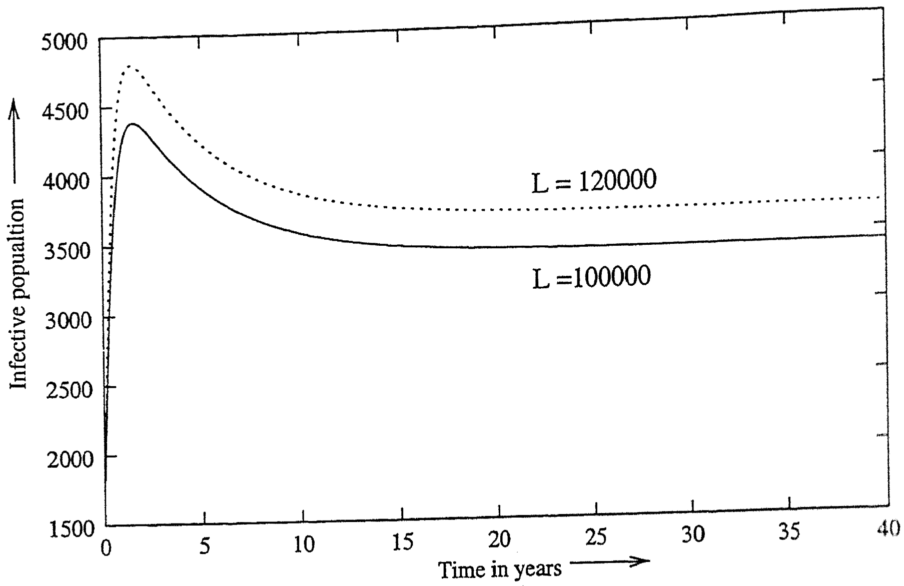


Figure 2.4: Variation of infective population with time for different carrying capacities of carrier population.

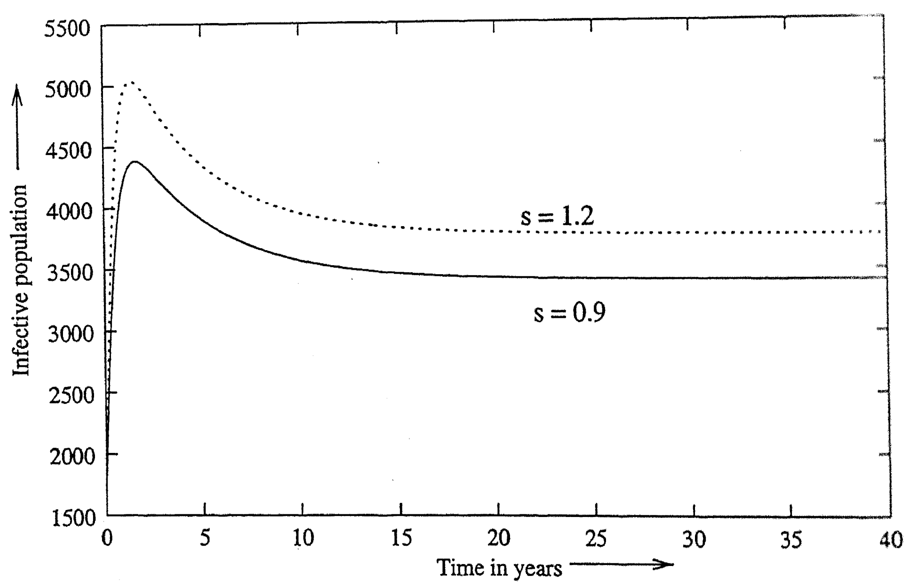


Figure 2.5: Variation of infective population density with time for different intrinsic growth rates of carrier population.

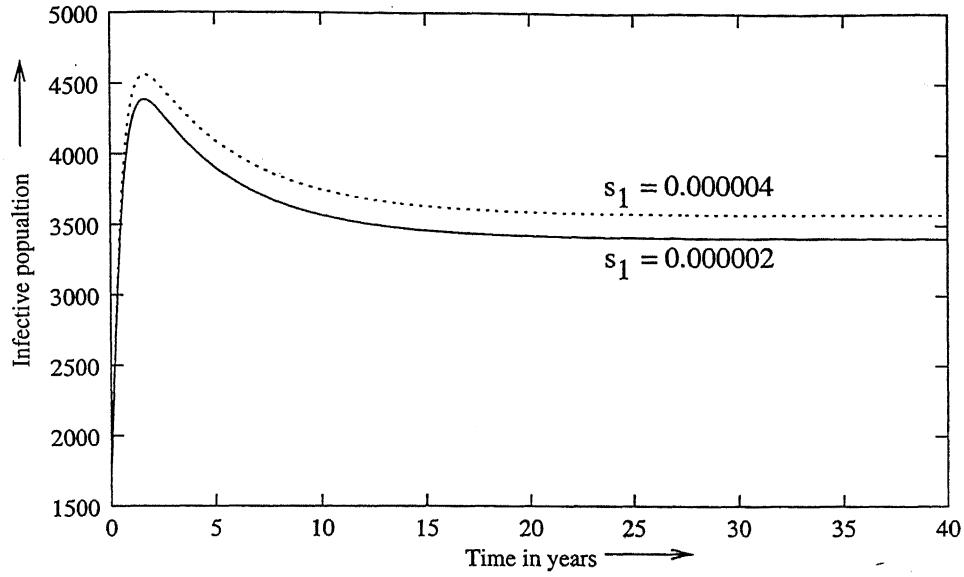


Figure 2.6: Variation of infective population with time for different growth rate coefficients of carrier population due to the cumulative environmental discharges.

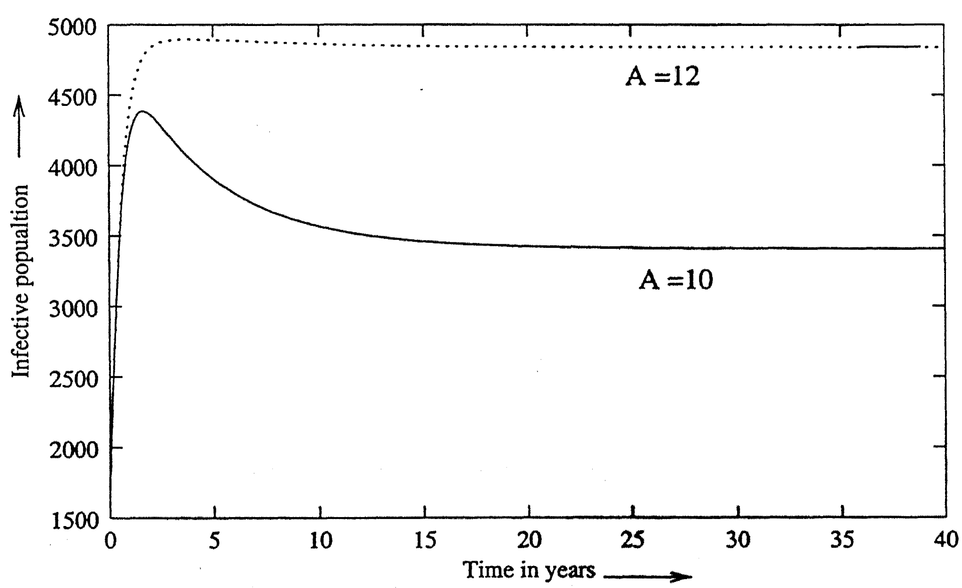


Figure 2.7: Variation of infective population with time for different immigration rates of human population.

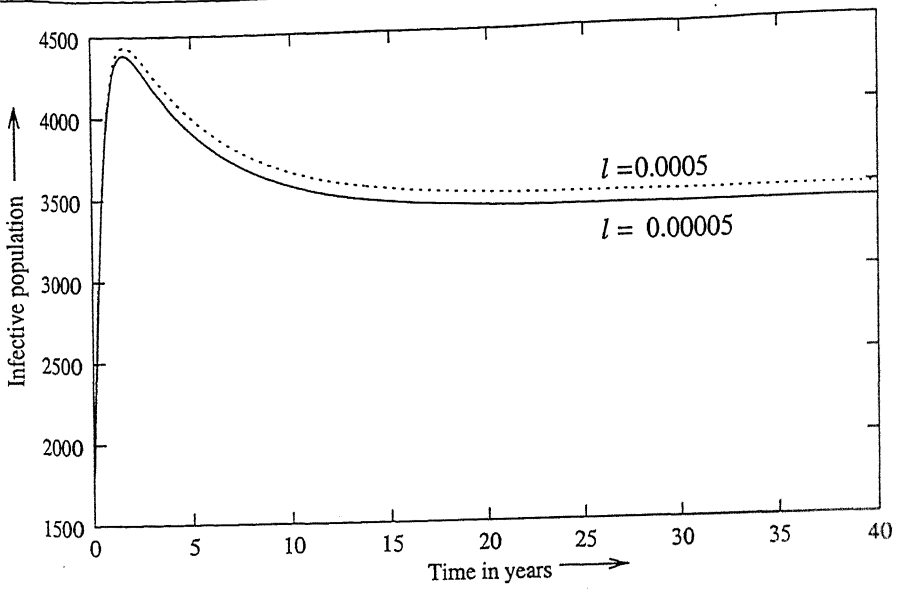


Figure 2.8: Variation of infective population with time for different l.

We see from this figure that for any initial starting condition, the solution curves tend to the equilibrium P_3. Hence we infer that the system (2.6) is globally stable about this endemic equilibrium point $P_3(\hat{Y},\ \hat{N},\ \hat{C},\ \hat{E})$ under local stability conditions for the set of parameters considered, provided that we start away from other equilibria. In Figs. 2.3-2.8, the effects of various parameters, i.e. Q_0, L, s, s_1, and l on the infective population have been shown. It is noted from these figures that as these parameters increase, the population of infectives increases. These results imply that as the carrier population increases due to household discharges, the spread of the disease increases and it becomes more endemic. Also an increase in population density due to immigration enhances further the spread of infectious diseases.

2.3 SIS Model with Logistic Population Growth

In this section, an SIS model with logistic growth of human population is considered so that both the birth as well as the death rates are density dependent in such a manner that the birth rate decreases and death rate increases as the population density increases towards its carrying capacity (Gao and Hethcote 1992). Keeping in view the consideration

of the previous sections, a mathematical model is proposed as follows:

$$
\begin{aligned}
\dot{X} &= \left[b - a\,\frac{rN}{K}\right] N - \left[d + (1-a)\,\frac{rN}{K}\right] X - \beta\, XY - \lambda\, XC + \nu Y,\\
\dot{Y} &= \beta\, XY + \lambda\, XC - \left[\nu + \alpha + d + (1-a)\,\frac{rN}{K}\right] Y,\\
\dot{N} &= r\left[1 - \frac{N}{K}\right] N - \alpha\, Y, \qquad\qquad (2.10)\\
\dot{C} &= sC(1 - \frac{C}{L}) - \delta\, C + s_1\, EC,\\
\dot{E} &= Q(N) - \delta_0\, E = Q_0 + lN - \delta_0 E,\\
X &+ Y = N, \quad s > \delta, \quad 0 \le a \le 1,
\end{aligned}
$$

$X(0) = X_0 > 0,\ Y(0) = Y_0 \ge 0,\ N(0) = N_0 > 0,\ C(0) = C_0 \ge 0$ and $E(0) = E_0 > 0$.

Here b and d are the natural birth and death rates; $r = b - d > 0$ is the growth rate constant; K is the carrying capacity of the human population density in the natural environment. All other parameters are as defined in the previous section.

For $0 < a < 1$, the birth rate decreases and the death rate increases as N increases to its carrying capacity K. When $a = 1$, the model could be called simply a logistic birth model as all of the restricted growth is due to a decreasing birth rate and the death rate is constant. Similarly, when $a = 0$, it could be called a logistic death model as all of the restricted growth is due to an increasing death rate and the birth rate is constant. It is easy to note that the above model is well-posed in the region of attraction T_1 given by,

$$T_1 = \left\{(Y, N, C, E) : 0 \le Y \le N \le K,\ 0 \le C \le \frac{L}{s}\left(s - \delta + s_1\,\frac{Q(K)}{\delta_0}\right), 0 \le\ E \le \frac{Q(K)}{\delta_0}\right\}.$$

2.3.1 Case I: Q is a Constant Q_a

Similarly to Section 2.2 here also it is sufficient to consider the following subsystem of system (2.10).

$$
\begin{aligned}
\dot{Y} &= \beta\,(N - Y)\,Y + \lambda\,(N - Y)\,C_m - \left[\nu + \alpha + d + (1-a)\,\frac{rN}{K}\right] Y,\\
\dot{N} &= r\left[1 - \frac{N}{K}\right] N - \alpha\, Y, \qquad\qquad (2.11)
\end{aligned}
$$

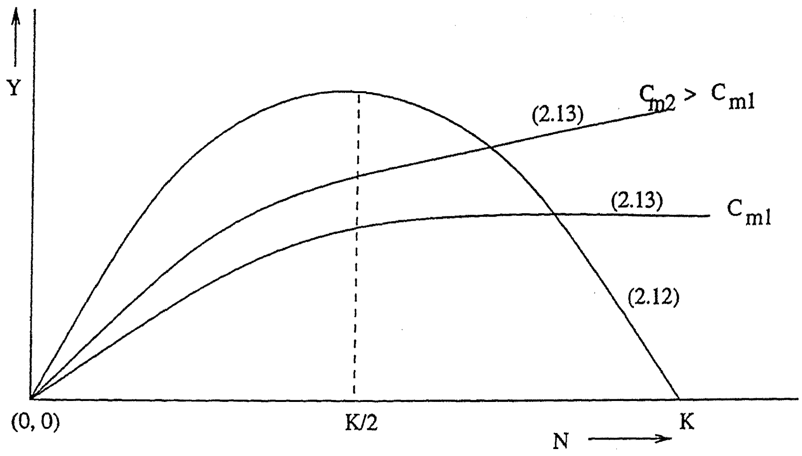


Figure 2.9: Existence of equilibrium point.

where $C_m = \frac{L}{s}\{s - \delta + s_1 \frac{Q_a}{\delta_0}\}$, which increases as the household discharge rate Q_a increases.

The result of an equilibrium analysis is stated in the following theorem.

THEOREM 2.5 There exist the following two equilibria, namely (i) $E_1(0,0)$ and (ii) $E_2(\hat{Y},\ \hat{N})$, which exists if $\nu + \alpha + d > \frac{\alpha - r}{r}\lambda C_m$.

Proof: Existence of E_1 is obvious. The existence of second equilibrium point is shown as follows:

Setting the right hand side of (2.11) to zero, we get

$$Y = \frac{r}{\alpha}(1 - \frac{N}{K})N, \tag{2.12}$$

$$\beta Y^2 - [\{\beta - (1-a)\frac{r}{K}\}N - (\nu + \alpha + d + \lambda C_m)]Y - \lambda C_m N = 0. \tag{2.13}$$

It may be pointed out that in the N-Y plane (2.12) gives a parabola with vertex $(\frac{K}{2}, \frac{rK}{4\alpha})$ and passing through the points (0,0) and $(K, 0)$, while (2.13) gives a hyperbola with a branch in the first and fourth quadrants and passing through (0,0). The two curves will intersect at a point $(\hat{Y}, \hat{N})$ provided the slope of parabola at (0,0) is more than that of the hyperbola branch in first quadrant at (0,0), i.e. slope of (2.13) at (0,0) is less than that of slope of (2.12) at (0,0),

$$\nu + \alpha + d > \frac{\alpha - r}{r}\lambda C_m \tag{2.14}$$

Thus the condition for the existence of second equilibrium point $(\hat{Y}, \hat{N})$, is proved.

Remark 1: From (2.13), $\left(\frac{dY}{dN}\right)_{(0,0)} = \frac{\lambda C_m}{\nu+\alpha+d+\lambda C_m} > 0,$ (2.15)

which increases as C_m increases.

Remark 2: It is also noted from (2.13) that $\frac{d\hat{Y}}{dC_m} > 0$ for $\hat{N} \geq \frac{K}{2}$, which implies that equilibrium infective density increases as C_m increases.

Remark 3: It is noted that if $\beta = (1-a)\frac{r}{K}$, then (2.13) gives a parabola and in this case also there exists a unique positive root $\hat{N}$ in $(0, K)$.

2.3.1.1 Stability Analysis

Now we present the stability analysis of these equilibria. The local stability results are stated in the following theorem.

THEOREM 2.6 The equilibrium $E_1(0,0)$ is unstable and the equilibrium $E_2(\hat{Y}, \hat{N})$ is locally asymptotically stable provided

$$\left(\beta\hat{Y} + \frac{\lambda\hat{N}C_m}{\hat{Y}}\right)\frac{r}{K}(2\hat{N} - K) + \alpha\{\beta - (1-a)\frac{r}{K}\}\hat{Y} + \lambda C_m\alpha > 0.$$

Proof: The variational matrix M_1 at $E_1(0,0)$ corresponding to the system of equations (2.11) is given by

$$M_1 = \begin{pmatrix} -\lambda\, C_m - (\nu + \alpha + d) & \lambda\, C_m \\ -\alpha & r \end{pmatrix}.$$

Since one eigenvalue of M_1 is positive, E_1 is unstable.

The variational matrix $\hat{M}$ at $E_2(\hat{Y},\ \hat{N})$ corresponding to system of equations (2.11) is given by

$$\hat{M} = \begin{pmatrix} -\left(\beta\,\hat{Y} + \frac{\lambda\,\hat{N}\,C_m}{\hat{Y}}\right) & \beta\,\hat{Y} + \lambda\,C_m - (1-a)\,r\,\frac{\hat{Y}}{K} \\ -\alpha & \frac{r}{K}(K - 2\hat{N}) \end{pmatrix},$$

The characteristic polynomial is given by

$$\psi^2 + \{\beta\,\hat{Y} + \frac{\lambda\,\hat{N}\,C_m}{\hat{Y}} + \frac{r}{K}(2\hat{N} - K)\}\psi + \left(\beta\hat{Y} + \frac{\lambda\hat{N}C_m}{\hat{Y}}\right)\frac{r}{K}(2\hat{N} - K)$$

$$+\alpha\{\beta\,\hat{Y}+\lambda\,C_m-(1-a)\frac{r\,\hat{Y}}{K}\}=0. \tag{2.16}$$

In order that the above quadratic has roots which have negative real parts, it is necessary that

$$\beta\,\hat{Y}+\frac{\lambda\,\hat{N}\,C_m}{\hat{Y}}+\frac{r}{K}(2\hat{N}-K)>0, \tag{2.17}$$

$$\text{and}\quad \left(\beta\hat{Y}+\frac{\lambda\hat{N}C_m}{\hat{Y}}\right)\frac{r}{K}(2\hat{N}-K)+\alpha\{\beta-(1-a)\,\frac{r}{K}\}\hat{Y}+\alpha\,\lambda C_m>0, \tag{2.18}$$

The first inequality is obviously true in view of the equilibrium conditions i.e. (2.12) and (2.13). Thus only the second inequality gives the condition for linear stability of $E_2(\hat{Y},\hat{N})$. Hence the theorem.

Remark: It may be noted that the inequality (2.18) is satisfied for $\hat{N}\geq\frac{K}{2}$ and $\beta>\frac{(1-a)r}{K}$. The condition $\hat{N}\geq\frac{K}{2}$ is also compatible with the condition of $\frac{d\hat{Y}}{dC_m}>0$. Hence from now onward we assume the above mentioned condition.

Nonlinear Analysis and Simulation

Using the same Liapunov function as in (2.5) and the system (2.11), we get

$$\dot{V}=-(\beta+\frac{\lambda NC_m}{Y\hat{Y}})(Y-\hat{Y})^2-k_1\frac{r}{K}(N+\hat{N}-K)(N-\hat{N})^2$$

$$-\{k_1\alpha-\frac{\lambda C_m}{\hat{Y}}-(\beta-\overline{1-a}\,\frac{r}{K})\}(Y-\hat{Y})(N-\hat{N}).$$

After choosing $k_1=\frac{1}{\alpha}\{\frac{\lambda C_m}{\hat{Y}}+\beta-(1-a)\frac{r}{K}\}$, we note that $\dot{V}$ is negative definite in the region $\frac{K}{2}<N\leq K$ such that $\frac{K}{2}<\hat{N}\leq K$, where it is assumed that $\beta>(1-a)\frac{r}{K}$. Hence $E_2(\hat{Y},\hat{N})$ is globally asymptotically stable in a subregion of T_1.

Also we note that the model (2.11) is globally stable when the disease related death rate α is zero and the system (2.11) is bounded by its corresponding system (2.11) with $\alpha=0$. So using comparison theorems (Lakshmikantham and Leela 1969), it is concluded that solution of the system (2.11) is bounded by the solution of the system (2.11) with $\alpha=0$. Therefore, it is speculated that the nontrivial equilibrium point E_2 of model (2.11) may be globally stable for $\hat{N}\geq\frac{K}{2}$. To illustrate this and to see the effects of various parameters on the spread of the disease, the system (2.11) is integrated by the

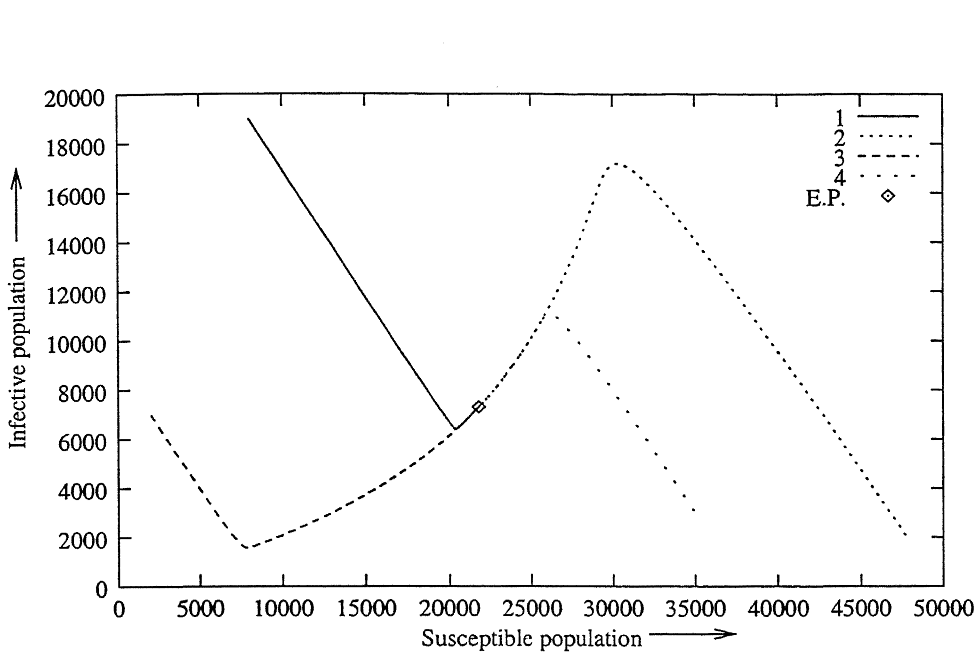


Figure 2.10: Variation of infective population with susceptible population.

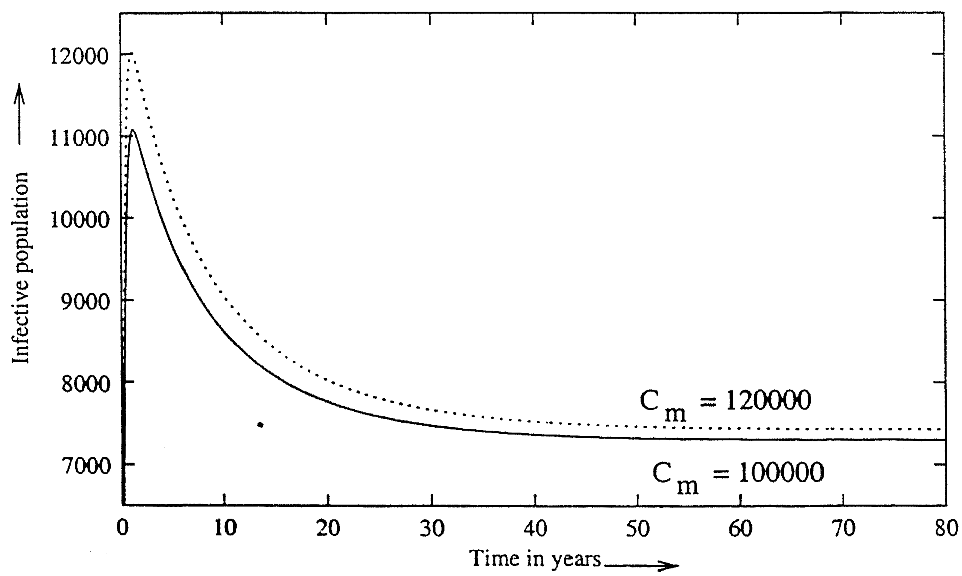


Figure 2.11: Variation of infective population with time for different C_m.

fourth order Runge-Kutta method using following set of parameters in the simulation, which satisfies the local stability condition (2.17) of equilibrium E_2.

$$\beta = 0.00000031, \ \lambda = 0.000000021, \nu = 0.012, \ \alpha = 0.0005,$$

$$a = 0.3, \ d = 0.0004, \ r = 0.0003, \ K = 50000, \ C_m = 100000.$$

The equilibrium values of $\hat{Y}$ and $\hat{N}$ are obtained as $\hat{Y} = 7299.645$ and $\hat{N} = 29087.309$. Simulation is performed for different initial positions 1, 2, 3, 4 as shown in Fig. 2.10. In this figure, the infected population is plotted against the susceptible population. From the solution curves, we observe that the system is globally stable for this set of parameters, provided that we start away from other equilibria. In Fig. 2.11, the infective population is plotted against time for different C_m and from this we observe that the infective population increases as C_m increases.

2.3.2 Case II: Q is a Variable

In this case we consider the following equivalent system of system (2.10) (using $X + Y = N$),

$$\begin{aligned}
\dot{Y} &= \beta(N-Y)\,Y + \lambda(N-Y)\,C - \left[\nu + \alpha + d + (1-a)r\frac{N}{K}\right] Y, \\
\dot{N} &= r\left(1 - \frac{N}{K}\right) N - \alpha\, Y, \\
\dot{C} &= sC\left(1 - \frac{C}{L}\right) - \delta\, C + s_1\, EC, \\
\dot{E} &= Q(N) - \delta_0\, E = Q_0 + lN - \delta_0 E.
\end{aligned} \tag{2.19}$$

The result of equilibrium analysis is stated in the following theorem.

THEOREM 2.7 There exist the following five equilibria, namely
(i) $\bar{E}_1(0, \ 0, \ 0, \frac{Q_0}{\delta_0})$, *(ii)* $\bar{E}_2(0, \ K, \ 0, \ \frac{Q_0}{\delta_0})$, *(iii)* $\bar{E}_3(Y^*, \ N^*, \ 0, \ E^*)$, *which exists if* $\beta K > (1-a)r + \nu + \alpha + d$,

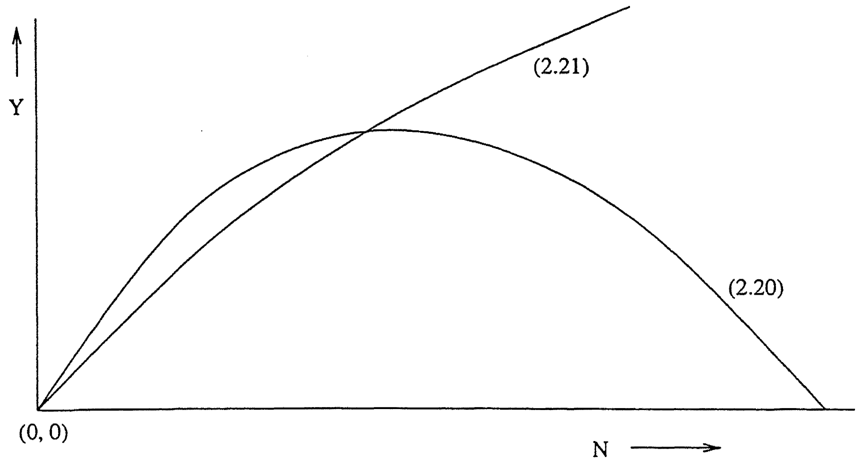


Figure 2.12: Existence of equilibrium point.

where $N^* = \dfrac{-\{\beta(1-\frac{r}{\alpha}) - (1-a)\frac{r}{K}\} + \sqrt{\{\beta(1-\frac{r}{\alpha}) - (1-a)\frac{r}{K}\}^2 + 4\,\frac{\beta r}{\alpha K}(\nu+\alpha+d)}}{2\frac{\beta r}{\alpha K}}$,

$$Y^* = \frac{r}{\alpha}\left(1 - \frac{N^*}{K}\right)N^* > 0, \qquad E^* = \frac{Q(N^*)}{\delta_0},$$

(iv) $\bar{E}_4(0,\ 0,\ \bar{C},\ \bar{E})$, *where* $\bar{C} = \frac{L}{s}\{s - \delta + s_1\frac{Q_0}{\delta_0}\}$, $\bar{E} = \frac{Q_0}{\delta_0}$ *and*

(v) $\bar{E}_5(\hat{Y},\ \hat{N},\ \hat{C},\ \hat{E})$, *which exists if* $(\nu+\alpha+d) > \frac{\lambda L}{s}\frac{(\alpha-r)}{r}\left(s - \delta + s_1\frac{Q_0}{\delta_0}\right)$.

Proof: The existence of the first four equilibria is obvious. The existence of the fifth equilibrium $\bar{E}_5$ is shown as follows. Setting the right hand side of system (2.19) to zero and simplifying we get

$$Y = \frac{r}{\alpha}\,(1 - \frac{N}{K})N, \tag{2.20}$$

$$\beta Y^2 - [\{\beta - (1-a)\frac{r}{K}\}N - (\nu+\alpha+d+\lambda C)]Y - \lambda CN = 0, \tag{2.21}$$

where $C = \frac{L}{s}\{s - \delta + s_1\frac{Q_0 + lN}{\delta_0}\}$.

As before, we see that in N-Y plane (2.20) is a parabola and (2.21) is a hyperbola unless $\beta = (1-a)\frac{r}{K}$ when it is a parabola passing through origin and a branch in the first quadrant for $\lambda > 0$.

From (2.21),

the slope $\left(\dfrac{dY}{dN}\right) = Y\left[\dfrac{k_1 Y + \lambda C + \frac{\lambda L}{s}\frac{l\,s_1}{\delta_0}(N-Y)}{(\beta Y^2 + \lambda CN)}\right] > 0$, for $Y > 0,\ N > 0$,

where $k_1 = \beta - (1-a)\frac{r}{K}$, assumed positive.

Also $\left(\dfrac{dY}{dN}\right)_{(0,0)} = \dfrac{\frac{\lambda L}{s}(s-\delta+\frac{s_1}{\delta_0}Q_0)}{\nu+\alpha+d+\frac{\lambda L}{s}(s-\delta)+\frac{\lambda s_1}{\delta_0}Q_0} > 0.$

Using these aspects and plotting (2.20) and (2.21) in the first quadrant (Fig. 2.12), we see that for the existence of nontrivial $\hat{Y}$ and $\hat{N}$, the slope of (2.20) at $(0,0)$ must be greater than the slope of (2.21) at $(0,0)$, i.e.

$$\frac{r}{\alpha} > \frac{\frac{\lambda L}{s}\left(s-\delta+\frac{s_1}{\delta_0}Q_0\right)}{\nu+\alpha+d+\frac{\lambda L}{s}\left(s-\delta+\frac{s_1}{\delta_0}Q_0\right)}$$

$$\text{or} \quad (\nu+\alpha+d) > \frac{\lambda L}{s}\frac{(\alpha-r)}{r}\left(s-\delta+s_1\frac{Q_0}{\delta_0}\right), \tag{2.22}$$

which is same as (2.15) for $Q_0 = Q_a$. Thus, after knowing $\hat{Y}$ and $\hat{N}$, corresponding values of $\hat{C}$ and $\hat{E}$ can be calculated as follows; $\hat{C} = \frac{L}{s}\{s-\delta+s_1\hat{E}\}$ and $\hat{E} = \frac{Q_0+l\hat{N}}{\delta_0}$. Here the inequality (2.22) is the ufficient ondition for existence of the fifth equilibrium point $\bar{E}_5(\hat{Y},\hat{N},\hat{C},\hat{E})$.

Remark: In both the cases when carrier population is absent, for disease to grow we must have a threshold condition as $\frac{\beta K-(1-a)r}{\nu+\alpha+d} > 1$, which is same as mentioned in Gao and Hethcote (1992).

2.3.2.1 Stability Analysis

Now we discuss the linear stability of these equilibria and nonlinear stability only of the nontrivial equilibrium $\bar{E}_5$.

The local stability results of all equilibria are stated in the following theorem.

THEOREM 2.8 The equilibria $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ are unstable. The fourth equilibria $\bar{E}_4$ is stable if

$$\lambda\bar{C}+\nu+\alpha+d > r \quad \textit{and} \quad \frac{(\alpha-r)}{r}\lambda\bar{C} > (\nu+\alpha+d),$$

otherwise if

$$\lambda\bar{C}+\nu+\alpha+d < r \quad \textit{or} \quad \frac{(\alpha-r)}{r}\lambda\bar{C} < (\nu+\alpha+d),$$

it is unstable and the fifth equilibrium $\bar{E}_5$ exists. The fifth equilibrium is locally asymptotically stable provided $\begin{vmatrix} a_3 & a_1 \\ 1 & a_2 \end{vmatrix} > 0$ *and* $\begin{vmatrix} a_3 & a_1 & 0 \\ 1 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0$, *where* a_0, a_1, a_2, *and* a_3 *are given in the proof of the theorem.*

Proof: The variational matrices M_1, M_2, M_3, M_4 and M_5 corresponding to system (2.19) at equilibrium points $\bar{E}_1$, $\bar{E}_2$, $\bar{E}_3, \bar{E}_4$ and $\bar{E}_5$ respectively are given by,

$$M_1 = \begin{pmatrix} -(\nu+\alpha+d) & 0 & 0 & 0 \\ -\alpha & r & 0 & 0 \\ 0 & 0 & s-\delta+s_1\frac{Q_0}{\delta_0} & 0 \\ 0 & l & 0 & -\delta_0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} \beta K - \{\nu+\alpha+d+(1-a)r\} & 0 & \lambda K & 0 \\ -\alpha & r & 0 & 0 \\ 0 & 0 & s-\delta+s_1\frac{Q(K)}{\delta_0} & 0 \\ 0 & l & 0 & -\delta_0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} m'_{11} & m'_{12} & \lambda N^*\{1-\frac{r}{\alpha}(1-\frac{N^*}{K})\} & 0 \\ -\alpha & r-\frac{2rN^*}{K} & 0 & 0 \\ 0 & 0 & s-\delta+s_1\frac{Q(N^*)}{\delta_0} & 0 \\ 0 & l & 0 & -\delta_0 \end{pmatrix},$$

where $m'_{11} = -\beta\frac{r}{\alpha}N^*(1-\frac{N^*}{K})$ and $m'_{12} = \{\beta-(1-a)\frac{r}{K}\}\frac{r}{\alpha}(1-\frac{N^*}{K})N^*$,

$$M_4 = \begin{pmatrix} -(\lambda\bar{C}+\nu+\alpha+d) & \lambda\bar{C} & 0 & 0 \\ -\alpha & r & 0 & 0 \\ 0 & 0 & -\frac{s\bar{C}}{L} & s_1\bar{C} \\ 0 & l & 0 & -\delta_0 \end{pmatrix}$$

and

$$M_5 = \begin{pmatrix} -(\beta\hat{Y}+\frac{\lambda\hat{N}\hat{C}}{\hat{Y}}) & \{\beta-(1-a)\frac{r}{K}\}\hat{Y}+\lambda\hat{C} & \lambda(\hat{N}-\hat{Y}) & 0 \\ -\alpha & r-\frac{2r\hat{N}}{K} & 0 & 0 \\ 0 & 0 & -\frac{s\hat{C}}{L} & s_1\hat{C} \\ 0 & l & 0 & -\delta_0 \end{pmatrix}$$

respectively. Since the matrices M_1, M_2 and M_3 have positive eigen values so the equilibrium points corresponding to these matrices are unstable.

The characteristic polynomial corresponding to matrix M_4 is

$$(\psi+\delta_0)\left(\psi+\frac{s\bar{C}}{L}\right)\{\psi^2+(\lambda\bar{C}+\nu+\alpha+d-r)\psi-r(\lambda\bar{C}+\nu+\alpha+d)+\alpha\lambda\bar{C}\}=0$$

Clearly two roots are negative. Using the Routh-Hurwitz criteria, this equilibrium point is locally asymptotically stable if the following conditions are satisfied

$$\lambda\bar{C}+\nu+\alpha+d > r \quad \text{and} \quad \frac{(\alpha-r)}{r}\lambda\bar{C} > (\nu+\alpha+d), \tag{2.23}$$

otherwise if

$$\lambda\bar{C}+\nu+\alpha+d < r \quad \text{or} \quad \frac{(\alpha-r)}{r}\lambda\bar{C} < (\nu+\alpha+d),$$

it is unstable and the fifth equilibrium exists as mentioned earlier in condition (2.22). It is clear from above that the second inequality in (2.23) may not be satisfied even if the first is satisfied, but when the first inequality is not satisfied, the second will not be satisfied. So violation of any one of the above inequalities gives existence of the fifth equilibrium.

The characteristic polynomial corresponding to matrix M_5 is

$$\psi^4+a_3\psi^3+a_2\psi^2+a_1\psi+a_0=0,$$

where

$$\begin{aligned}
a_3 &= \frac{\lambda(\hat{N}-\hat{Y})\hat{C}}{\hat{Y}}+(\nu+d)\frac{\hat{Y}}{(\hat{N}-\hat{Y})}+\frac{\alpha\hat{Y}^2}{\hat{N}(\hat{N}-\hat{Y})}+(1-a)\frac{r\hat{N}\hat{Y}}{K(\hat{N}-\hat{Y})}\\
&\quad+\frac{r\hat{N}}{K}+\frac{s}{L}\hat{C}+\delta_0 > 0,\\
a_2 &= \left(\beta\hat{Y}+\frac{\lambda\hat{N}\hat{C}}{\hat{Y}}\right)\left[\{\frac{r}{K}(2\hat{N}-K)\}+\frac{s}{L}\hat{C}+\delta_0\right]+\frac{r}{K}\left(2\hat{N}-K\right)\left\{\frac{s}{L}\hat{C}+\delta_0\right\}\\
&\quad+\frac{s}{L}\hat{C}\delta_0+\alpha\left[\left\{\beta-(1-a)\frac{r}{K}\right\}\hat{Y}+\lambda\hat{C}\right],\\
a_1 &= \left(\beta\hat{Y}+\frac{\lambda\hat{N}\hat{C}}{\hat{Y}}\right)\frac{r}{K}(2\hat{N}-K)\left(\frac{s}{L}\hat{C}+\delta_0\right)+\frac{r}{K}(2\hat{N}-K)\frac{s\hat{C}\delta_0}{L}\\
&\quad+\left(\beta\hat{Y}+\frac{\lambda\hat{N}\hat{C}}{\hat{Y}}\right)\frac{s\hat{C}\delta_0}{L}+\alpha[\{\beta-(1-a)\frac{r}{K}\}\hat{Y}+\lambda\hat{C}](\frac{s\hat{C}}{L}+\delta_0),\\
a_0 &= (\beta\hat{Y}+\frac{\lambda\hat{N}\hat{C}}{\hat{Y}})\frac{r}{K}(2\hat{N}-K)\frac{s\hat{C}\delta_0}{L}+\alpha\frac{s\hat{C}}{L}\delta_0[\{\beta-(1-a)\frac{r}{K}\}\hat{Y}+\lambda\hat{C}]\\
&\quad+\alpha\lambda(\hat{N}-\hat{Y})s_1\hat{C}l.
\end{aligned}$$

By the Routh-Hurwitz criteria, conditions for local stability of the system are

$$a_3 > 0, \quad \begin{vmatrix} a_3 & a_1 \\ 1 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_3 & a_1 & 0 \\ 1 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0 \text{ and } \begin{vmatrix} a_3 & a_1 & 0 & 0 \\ 1 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & 1 & a_2 & a_0 \end{vmatrix} > 0.$$

Clearly the first inequality is obvious. If the second and the third inequalities are satisfied, so is the fourth one. Hence the equilibrium point $\bar{E}_5$ is locally asymptotically stable if the second and the third inequalities are satisfied.

Remark: It is seen that second is satisfied for $\hat{N} \geq \frac{K}{2}$ and $\beta > \frac{(1-a)r}{K}$. So in this case only third inequality is the condition for local stability of $\bar{E}_5$.

Nonlinear Analysis and Simulation

As before we speculate that the nontrivial equilibrium point E_5 of the model (2.19) may be globally stable under the local stability conditions. To illustrate this and to see the effects of various parameters on the spread of the disease, the system (2.19) is integrated using the fourth order Runge-Kutta Method by taking $Q(N) = Q_0 + lN$ and using the following set of parameters in the simulation, which satisfies the local stability condition mentioned above.

$$\beta = 0.00000031, \quad \lambda = 0.000000021, \quad \nu = 0.012, \quad \alpha = 0.0005,$$

$$\delta = 0.6, \quad \delta_0 = 0.001, \quad r = 0.0003, \quad d = 0.0004, \; a = 0.3, \; K = 50000,$$

$$s = 0.9, \quad Q_0 = 20, \quad s_1 = 0.000002, \; l = 0.00005 \; \text{ and } \; L = 100000.$$

All the parameters are in units of per day except the carrying capacity L, which has the same dimension as C.

The equilibrium values of $\hat{Y}$, $\hat{N}$, $\hat{C}$ and $\hat{E}$ have been found as

$$\hat{Y} = 6121.797, \quad \hat{N} = 35716.819, \quad \hat{C} = 38174.626, \quad \hat{E} = 21785.839.$$

In Fig. 2.13, the infected population is plotted against the susceptible population and from the solution curves, it is concluded that the system appears to be globally stable for this set of parameters. In Figs. 2.14-2.19 the effects of various parameters, i.e Q_0, L, s, s_1, r and l on the infective population have been shown. It is noted from these figures that as these parameter values increase, the infective population increases and we have similar conclusions regarding the spread of the infectious disease as discussed earlier.

Remark: When $K \to \infty$, the model (2.10) coincides with the model with exponential growth of the population, though here it is not considered explicitly.

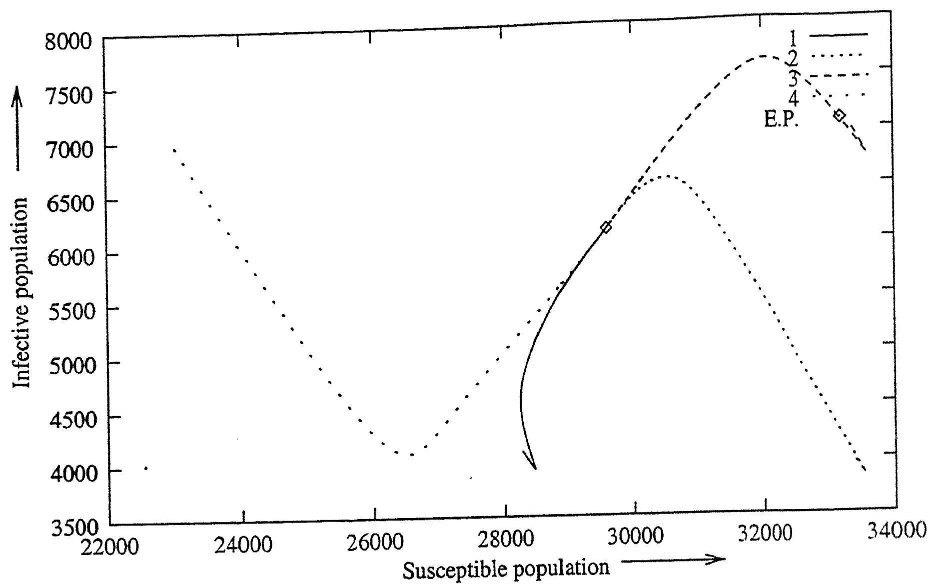


Figure 2.13: Variation of infective population with susceptible population.

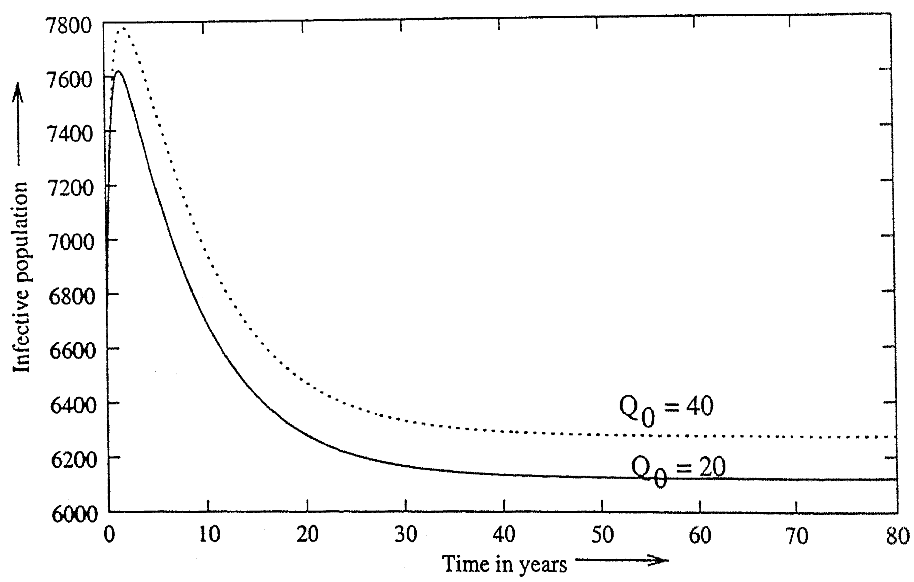


Figure 2.14: Variation of infective population with time for different cumulative environmental discharge rates.

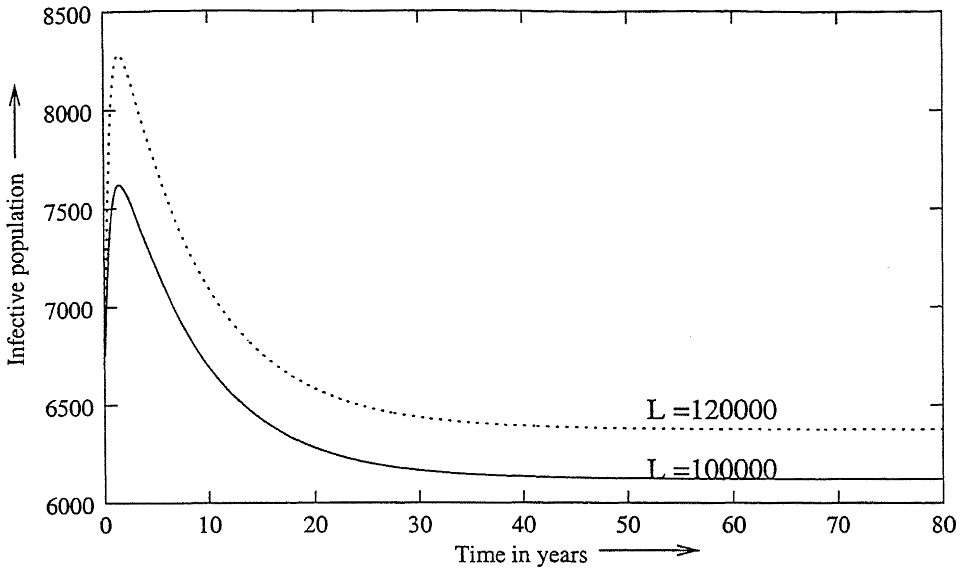


Figure 2.15: Variation of infective population with time for different carrying capacities of carrier population.

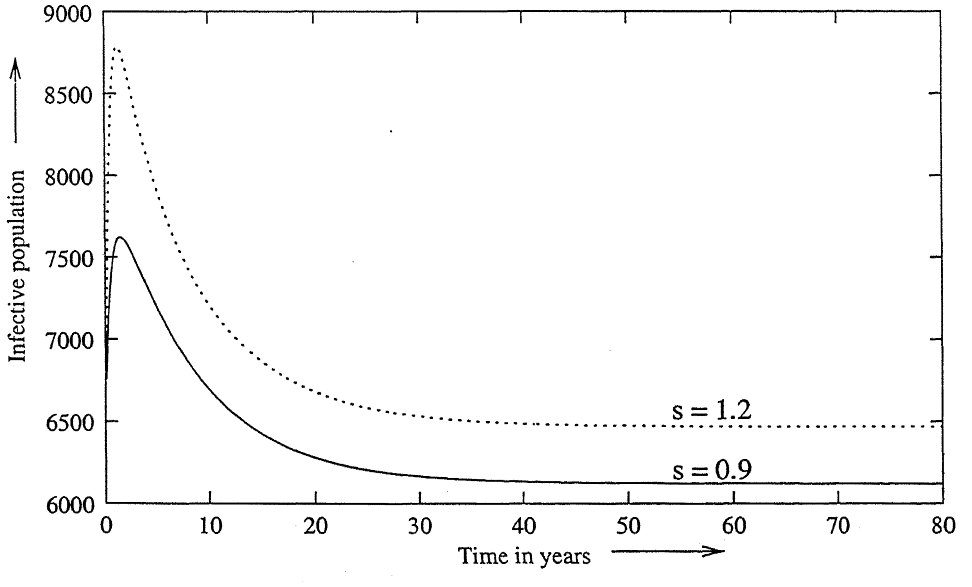


Figure 2.16: Variation of infective population with time for different intrinsic growth rates of carrier population.

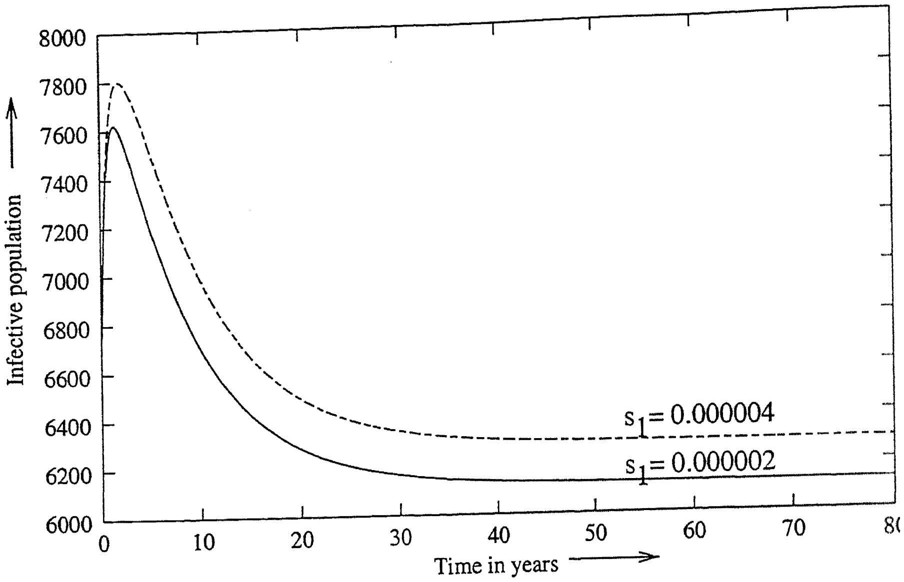


Figure 2.17: Variation of infective population with time for different growth rate coefficients of carrier population due to the cumulative environmental discharges.

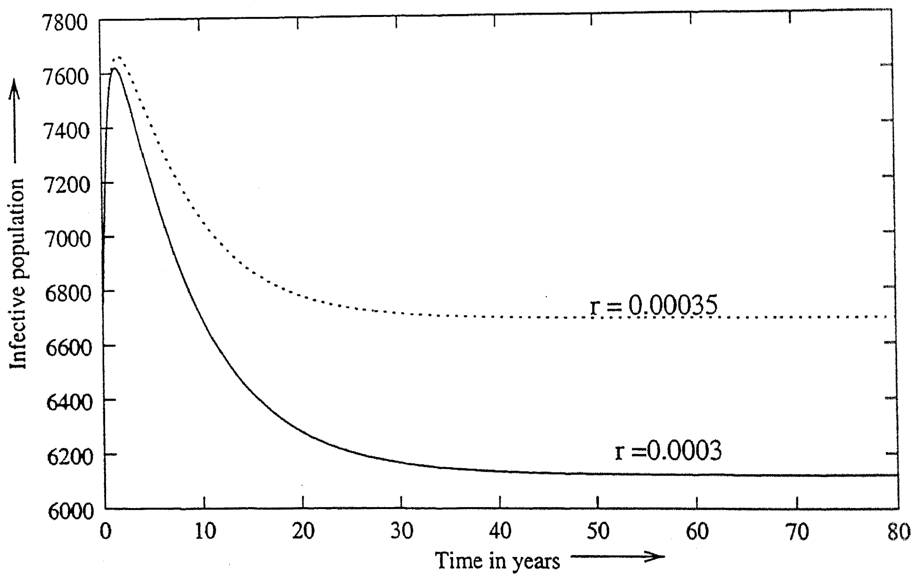


Figure 2.18: Variation of infective population with time for different intrinsic growth rates of human population.

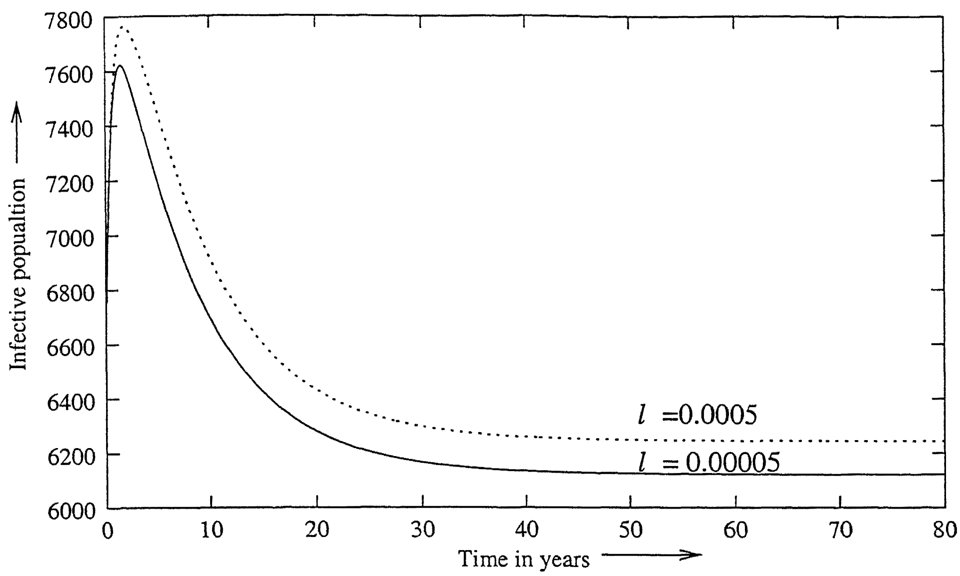


Figure 2.19: Variation of infective population with time for different l.

2.4 Conclusions

In this chapter SIS models for carrier dependent infectious diseases, like cholera, diarrhea, etc. caused by direct contact of susceptibles with infectives as well as by carriers are proposed and analyzed. The following two types of demographics are considered, (i) constant immigration, and (ii) logistic population growth. For both the models, equilibrium analysis is presented. It is shown that in the case of constant immigration the endemic equilibria is globally stable when the rate of cumulative density of household discharges is a constant. When it is a function of total human population, the endemic equilibria is again shown to be locally and globally stable under certain conditions. The later result is shown by computer simulation. But in the case of logistic population growth, the local stability of the nontrivial equilibria in both the cases is guaranteed only under certain conditions. By computer simulation it is shown that under the local stability conditions, the nontrivial equilibrium appears to be globally stable in both the cases. It is concluded from the analysis that if the growth of carrier population caused by conducive household discharges increases, the spread of the infectious disease increases. Also when the human population increases due to demographic changes, the infectious disease spreads even further and becomes more endemic.

Chapter 3

Modelling Bacterial Disease with Environmental and Demographic Effects

3.1 Introduction

The widespread occurrence and increasing incidence of various infectious diseases such as gonorrhea, tuberculosis, hepatitis, measles, influenza etc. are becoming major public health problems in most tropical countries and elsewhere (Bailey 1957, 1979). In general, spread of infectious diseases in human population depends upon various factors such as the numbers of infectives and susceptibles; modes of transmission (carriers, vectors, etc.); socio-economic factors; environmental, ecological and geographical and similar factors (Fuzzi et al. 1997, Dufour 1982). A detailed account of modelling and study of epidemic diseases can be found in literature in the form of lecture notes, monographs and similar survey. (Waltman 1974, Bailey 1975, 1982, Hethcote 1976, Hethcote et. al. 1981). The population biology of infectious diseases has also been presented by Anderson and May (1979). While several infectious diseases are spread by direct contact between susceptibles and infectives, there are some diseases (such as tuberculosis and typhoid) which are also transmitted to the human population indirectly by the flow of bacteria from infectives into the environment, for example by contaminated water used by susceptibles. Gonzalez-

Guzman (1989) analyzed an SIS model for the spread of typhoid by including the effect of indirect transmission with the flow of bacteria from infective in the environment. It is well known that bacteria grow and survive in large numbers in almost every environment on earth including polluted water in ponds, lakes, rivers, in acidic and alkaline waters, in the presence of toxic substances, and also in air and soil (Dufour 1982). Also when various kinds of household wastes are discharged into the environment, some disease causing bacteria find these discharges very conducive to their population growth. For example, cholera bacteria can survive by sheltering beneath the mucus outer coat of various algae and zooplankton. In the case of typhoid, the typhie bacteria multiply in milk products and other food wastes. Warmer water may increase algae blooms, helping vibrio cholera to multiply and perhaps even promote the emergence of new genetic strains. Thus increase in population of bacteria in the environment enhances the spread of infectious disease in human habitat.

Further the growth of human population in a habitats plays an important role in the spread of infectious diseases (Hethcote and Zhao, 1994, May and Anderson 1979). In general human populations vary and population change takes place in a habitat due to immigration, growth rate, etc.

In this chapter, therefore SIS models for the spread of infectious diseases are proposed and analyzed by considering environmental and demographic factors.

3.2 SIS Model with Immigration

We consider here, an SIS model with immigration. The disease is assumed to be spread directly by infectives as well as by the flow of bacteria into the environment from infectives. The total population density N(t) is divided into a susceptible class X(t) and an infective class Y(t). It is assumed that all susceptibles living in the habitat are affected by the bacteria population, whose density grows logistically with a given intrinsic growth rate and carrying capacity. The growth of the bacteria density is further assumed to increase as the cumulative density of discharges into the environment by the human

population increases. Keeping this in mind, a mathematical model is proposed as follows:

$$\begin{aligned}
\dot{X} &= A - dX - (\beta Y + \lambda B)X + \nu Y, \\
\dot{Y} &= (\beta Y + \lambda B)X - (\nu + \alpha + d)Y, \\
\dot{N} &= A - dN - \alpha Y, \\
\dot{B} &= sB\left(1 - \frac{B}{L}\right) + s_1 Y - s_0 B + \delta BE, \\
\dot{E} &= Q(N) - \delta_0 E, \\
X &+ Y = N,
\end{aligned} \tag{3.1}$$

$$X(0) = X_0 > 0, \;\; Y(0) = Y_0 \geq 0, \;\; N(0) = N_0 > 0, \;\; B(0) = B_0 \geq 0 \text{ and } E(0) = E_0 > 0.$$

Here $E(t)$ is the density of cumulative environmental discharges conducive to the growth of bacteria population; A is the immigration rate of human population assumed to be a constant; d is the natural death rate constant; β and λ are the transmission coefficients due to the infectives and bacteria population respectively; α is the disease related death rate constant; ν is the recovery rate constant; s is the intrinsic growth rate of the bacteria population; L is the carrying capacity of the bacteria population; s_0 is the death rate of bacteria due to control measures; s_1 is the rate of release of bacteria from the infective population in the environment; δ is the rate of growth of the bacteria population due to environmental discharges; $Q(N)$ (assumed to be increasing in N) is the cumulative rate of environmental discharges which may be population density dependent and δ_0 is its depletion rate coefficient. We note that $s > s_0$. In writing the model (3.1), we assume that new cases of disease occur at rates βXY and λXB due to the interaction of susceptibles with infectives and bacteria respectively (Anderson and May 1983).

The region of attraction corresponding to (3.1) is given by,

$$T = \left\{(Y, N, B, E) : 0 \leq Y \leq N \leq \frac{A}{d},\; 0 \leq B \leq B_{\max},\; 0 \leq E \leq \frac{Q(\frac{A}{d})}{\delta_0}\right\},$$

$$\text{where } B_{\max} = \frac{L}{2s}\left[\left\{s - s_0 + \delta\frac{Q(\frac{A}{d})}{\delta_0}\right\} + \sqrt{\left\{s - s_0 + \delta\frac{Q(\frac{A}{D})}{\delta_0}\right\}^2 + \frac{4ss_1A}{dL}}\;\right],$$

is positively invariant and all solutions starting in this region T stay in it. The continuity of the right hand sides of equations 3.1 and their derivatives implies that a unique solution

exists (Hale 1969). We analyze the model (3.1) in the following two cases:

(i) the cumulative rate of environmental discharges Q is a constant, and

(ii) the cumulative rate of environmental discharges Q is a function of total population density, which we consider as $Q = Q_0 + lN$, where Q_0 and l are constants.

3.2.1 Case I: Q is a Constant Q_a

Since $X+Y = N$, the system (3.1) can be studied by the following autonomous subsystem

$$\begin{aligned} \dot{Y} &= (\beta Y + \lambda B)(N - Y) - (\nu + \alpha + d)Y, \\ \dot{N} &= A - dN - \alpha Y, \\ \dot{B} &= sB\left(1 - \frac{B}{L}\right) + s_1 Y - s_0 B + \delta BE, \\ \dot{E} &= Q_a - \delta_0 E. \end{aligned} \tag{3.2}$$

It may be noted that $E \to \frac{Q_0}{\delta_0}$ as $t \to \infty$. Therefore it is reasonable to reformulate the above system by using the asymptotic value of E as follows:

$$\begin{aligned} \dot{Y} &= (\beta Y + \lambda B)(N - Y) - (\nu + \alpha + d)Y, \\ \dot{N} &= A - dN - \alpha Y, \\ \dot{B} &= sB\left(1 - \frac{B}{L}\right) + s_1 Y - s_0 B + \delta B\frac{Q_a}{\delta_0}. \end{aligned} \tag{3.3}$$

The result of an equilibrium analysis is stated in the following theorem.

THEOREM 3.1 For the system (3.3), there exist two equilibria, namely (i) $E_1\left(0, \frac{A}{d}, 0\right)$ and (ii) $E_2(\hat{Y}, \hat{N}, \hat{B})$.

Proof: Existence of the first equilibrium is obvious. Existence of the second is shown as follows. Setting the right hand side of (3.3) to zero, we get the following two equations,

$$\beta\left(1 + \frac{\alpha}{d}\right)Y^2 - \left\{\frac{\beta A}{d} - (\nu + \alpha + d) - \lambda\left(1 + \frac{\alpha}{d}\right)B\right\}Y - \frac{\lambda A}{d}B = 0 \tag{3.4}$$

$$\text{and } Y = \frac{1}{s_1}\left[\frac{s}{L}B^2 - \left\{s - s_0 + \delta\frac{Q_a}{\delta_0}\right\}B\right]. \tag{3.5}$$

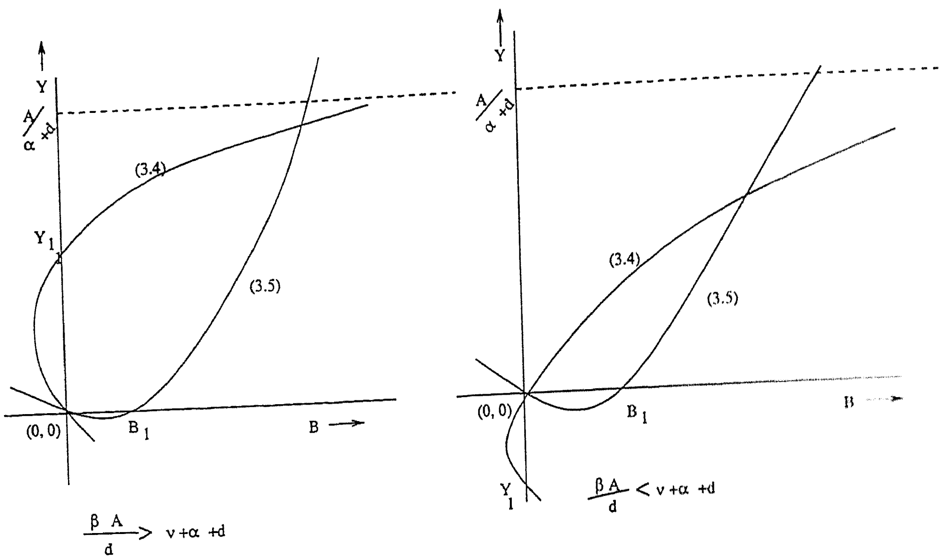


Figure 3.1: Existence of equilibrium point

From (3.4), we have

(i) when $B = 0$,

$$Y = 0 \text{ or } Y = \frac{\frac{\beta A}{d} - (\nu + \alpha + d)}{\beta(1 + \frac{\alpha}{d})} = Y_1 \text{ (say)} .$$

We see that $Y_1 < 0$, when $\frac{\beta A}{d} < (\nu + \alpha + d)$ and $Y_1 > 0$ when $\frac{\beta A}{d} > (\nu + \alpha + d)$.

(ii) $B \to \infty$ when $Y \to \dfrac{A}{\alpha + d}$

(iii) the slope at $(0, 0)$

$$\left(\frac{dY}{dB}\right) = \frac{\frac{-\lambda A}{d}}{\frac{\beta A}{d} - (\nu + \alpha + d)} \quad \begin{aligned} &< 0, \text{ if } \frac{\beta A}{d} > (\nu + \alpha + d) \\ &> 0, \text{ if } \frac{\beta A}{d} < (\nu + \alpha + d), \end{aligned}$$

also the slope at $(0, Y_1)$ is

$$\left(\frac{dY}{dB}\right) = \frac{\lambda(\nu + \alpha + d)}{\beta\{\frac{\beta A}{d} - (\nu + \alpha + d)\}} \quad \begin{aligned} &< 0, \text{ if } \frac{\beta A}{d} > (\nu + \alpha + d) \\ &> 0 \text{ , if } \frac{\beta A}{d} < (\nu + \alpha + d). \end{aligned}$$

From (3.5), we have

(i) when $Y = 0$, $B = 0$ or $B = \frac{L}{s}(s - s_0 + \delta \frac{Q_a}{\delta_0}) = B_1$ (say),

(ii) $\left(\dfrac{dY}{dB}\right)_{(0,0)} = -\dfrac{s - s_0 + \delta\frac{Q_a}{\delta_0}}{s_1} < 0$ and $\left(\dfrac{dY}{dB}\right)_{(B_1,0)} = \dfrac{s - s_0 + \delta\frac{Q_a}{\delta_0}}{s_1} > 0,$

(iii) $\dfrac{d^2Y}{dB^2} = \dfrac{2s}{s_1 L} > 0$, i.e slope increases with the increase of B.

Now plotting (3.4) and (3.5) in Fig. 3.1, we get an intersecting point $(\hat{Y}, \hat{B})$, where $\hat{Y} < \frac{A}{\alpha+d}$, irrespective of $\frac{\beta A}{d}$ greater or less than $(\nu + \alpha + d)$. Then $\hat{N}$ can be derived from $\hat{N} = \frac{A - \alpha\hat{Y}}{d}$, which is positive as $\hat{Y} < \frac{A}{\alpha}$.

3.2.1.1 Stability Analysis

Now we present the stability analysis of these equilibria. The local stability results are stated in the following theorem.

THEOREM 3.2 The equilibrium E_1 is unstable and the equilibrium E_2 is locally asymptotically stable.

Proof: The variational matrix M corresponding to system of equation (3.3) at (Y, N, B) is given by

$$M = \begin{pmatrix} \beta(N-Y) - (\beta Y + \lambda B + \nu + \alpha + d) & \beta Y + \lambda B & \lambda(N-Y) \\ -\alpha & -d & 0 \\ s_1 & 0 & s - s_0 + \delta\frac{Q_a}{\delta_0} - 2\frac{s}{L}B \end{pmatrix}.$$

Now, the variational matrix M_1 at equilibrium point $E_1\left(0, \frac{A}{d}, 0\right)$ is given by

$$M_1 = \begin{pmatrix} \beta\frac{A}{d} - (\nu + \alpha + d) & 0 & \lambda\frac{A}{d} \\ -\alpha & -d & 0 \\ s_1 & 0 & s - s_0 + \delta\frac{Q_a}{\delta_0} \end{pmatrix}.$$

We see that one eigenvalue corresponding to the above matrix is $-d$ and others are given by the following quadratic,

$$\psi^2 - \left(s - s_0 + \delta\frac{Q_a}{\delta_0} + \beta\frac{A}{d} - \overline{\nu + \alpha + d}\right)\psi + \left(s - s_0 + \delta\frac{Q_a}{\delta_0}\right)\left\{\beta\frac{A}{d} - \overline{\nu + \alpha + d}\right\} - \frac{\lambda A s_1}{d} = 0.$$

Clearly the equilibrium E_1 is locally asymptotically stable if

$$s - s_0 + \delta\frac{Q_a}{\delta_0} + \beta\frac{A}{d} - (\nu + \alpha + d) < 0, \quad \left(s - s_0 + \delta\frac{Q_a}{\delta_0}\right)\left\{\beta\frac{A}{d} - \overline{\nu + \alpha + d}\right\} - \frac{\lambda A s_1}{d} > 0.$$

We note that both the conditions are not satisfied simultaneously, so this equilibrium is unstable.

The variational matrix M_2 at equilibrium point $E_2(\hat{Y}, \hat{N}, \hat{B})$ is given by

$$M_2 = \begin{pmatrix} -(\beta\hat{Y} + \frac{\lambda\hat{B}\hat{N}}{\hat{Y}}) & \beta\hat{Y} + \lambda\hat{B} & \lambda(\hat{N} - \hat{Y}) \\ -\alpha & -d & 0 \\ s_1 & 0 & -(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}) \end{pmatrix}.$$

The characteristic polynomial is

$$\psi^3 + a_1\psi^2 + a_2\psi + a_3 = 0,$$

where

$$\begin{aligned} a_1 &= \beta\hat{Y} + \frac{\lambda\hat{B}\hat{N}}{\hat{Y}} + d + \frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}, \\ a_2 &= \left(\beta\hat{Y} + \frac{\lambda\hat{B}\hat{N}}{\hat{Y}}\right)\left(d + \frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right) + d\left(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right) + \alpha\left(\beta\hat{Y} + \lambda\hat{B}\right) - \lambda s_1\left(\hat{N} - \hat{Y}\right), \\ a_3 &= d\left(\beta\hat{Y} + \frac{\lambda\hat{B}\hat{N}}{\hat{Y}}\right)\left(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right) + \alpha\left(\beta\hat{Y} + \lambda\hat{B}\right)\left(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right) - \lambda s_1 d\left(\hat{N} - \hat{Y}\right). \end{aligned}$$

Here $a_1 > 0$, $a_3 > 0$ and also $a_1a_2 - a_3$ is positive. Hence E_2 is locally asymptotically stable if it exists.

Nonlinear Analysis and Simulation:

Before proceeding to computer simulation, we first analyze the model (3.3) when the disease related death rate α is zero. In this case the model (3.3) reduces to the following,

$$\begin{aligned} \dot{Y} &= (\beta Y + \lambda B)(N - Y) - (\nu + d)Y, \\ \dot{N} &= A - dN, \\ \dot{B} &= sB\left(1 - \frac{B}{L}\right) + s_1Y - s_0B + \delta B\frac{Q_a}{\delta_0}. \end{aligned} \tag{3.6}$$

Equating the right hand sides of the above equations to zero we get the following two equations for finding equilibrium points:

$$\beta Y^2 - \left(\beta\frac{A}{d} - \nu - \lambda B\right)Y - \lambda B\frac{A}{d} = 0, \tag{3.7}$$

$$sB\left(1 - \frac{B}{L}\right) + s_1Y - s_0B + \delta B\frac{Q_a}{\delta_0} = 0. \tag{3.8}$$

As before, plotting (3.7) and (3.8), we get two intersecting points $(0,0)$ and $(\hat{Y},\hat{B})$, correspondingly we get two equilibrium points $E_1^*\left(0,\frac{A}{d},0\right)$ and $E_2^*(\hat{Y},\hat{N},\hat{B})$.
By using the variational matrix method, it is easy to show that the equilibrium point E_1^* is unstable and E_2^* is locally asymptotically stable.
To study the global behavior of the system about the nontrivial equilibrium point E_2^*, we consider the following Liapunov function:

$$V=\frac{1}{2}(Y-\hat{Y})^2+\frac{1}{2}k_1(N-\hat{N})^2+\frac{1}{2}k_2(B-\hat{B})^2,$$

which gives

$$\begin{aligned}\dot{V} &= (Y-\hat{Y})\left\{\beta Y(N-\hat{N})+\beta\hat{N}(Y-\hat{Y})-\beta(Y+\hat{Y})(Y-\hat{Y})+\lambda B(N-\hat{N})\right.\\ &\quad \left.+\lambda\hat{N}(B-\hat{B})-\lambda B(Y-\hat{Y})-\lambda\hat{Y}(B-\hat{B})\right\}-(\nu+d)(Y-\hat{Y})^2-k_1d(N-\hat{N})^2\\ &\quad -k_2\left(\frac{s}{L}B+\frac{s_1\hat{Y}}{\hat{B}}\right)(B-\hat{B})^2+k_2s_1(Y-\hat{Y})(B-\hat{B})\\ &= -\left\{\beta Y+\lambda B+\frac{\lambda\hat{B}(\hat{N}-\hat{Y})}{\hat{Y}}\right\}(Y-\hat{Y})^2-k_1d(N-\hat{N})^2+(\beta Y+\lambda B)(Y-\hat{Y})(N-\hat{N})\\ &\quad +\left\{\lambda(\hat{N}-\hat{Y})+k_2s_1\right\}(Y-\hat{Y})(B-\hat{B})-k_2\left(\frac{s}{L}B+\frac{s_1\hat{Y}}{\hat{B}}\right)(B-\hat{B})^2.\end{aligned}$$

Using Sylvester's criteria, $\dot{V}$ will be negative definite provided the following inequalities are satisfied:

(i) $$4k_2\left\{\frac{\beta Y+\lambda B}{2}+\frac{\lambda\hat{B}(\hat{N}-\hat{Y})}{\hat{Y}}\right\}\left(\frac{s}{L}B+\frac{s_1\hat{Y}}{\hat{B}}\right)>\left\{\lambda(\hat{N}-\hat{Y})+k_2s_1\right\}^2$$

or $$2(\beta Y+\lambda B)k_2\left(\frac{s}{L}B+\frac{s_1\hat{Y}}{\hat{B}}\right)+4k_2\frac{s}{L}B\frac{\lambda\hat{B}(\hat{N}-\hat{Y})}{\hat{Y}}>\left\{\lambda(\hat{N}-\hat{Y})-k_2s_1\right\}^2 \tag{3.9}$$

Now choosing $k_2=\frac{\lambda(\hat{N}-\hat{Y})}{s_1}$, above inequality is satisfied provided $Y>0$, ($B=0$ or $B\neq 0$).

(ii) $$4\frac{(\beta Y+\lambda B)}{2}k_1d>(\beta Y+\lambda B)^2$$

which implies

$$2k_1d>(\beta Y+\lambda B) \text{ for } Y>0 \text{ \& } B>0. \tag{3.10}$$

Taking the maximum of the right hand side of the above inequality, we get

$$k_1 > \frac{\beta\frac{A}{d} + \lambda L}{2d}.$$

Thus for above choice of k_1 and k_2, $\dot{V}$ is negative definite in the interior of the region of attraction. Hence the equilibrium E_2^* of the system (3.6) is globally stable provided we start away from the disease free equlibrium.
Hence the model (3.3) is bounded by the system (3.6), which is globally stable. So using comparison theorems (Lakshmikantham and Leela 1969), the solution of (3.3) is bounded by the solution of (3.6). Hence we speculate that the nontrivial equilibrium point of (3.3) may be globally stable. To illustrate this, the system (3.3) is integrated using the fourth order Runge-Kutta Method and using the following set of parameters in the simulation (Greenhalgh 1990, 1992).

$$\beta = 0.00000031, \quad \lambda = 0.00000000021, \quad \nu = 0.012, \quad \alpha = 0.0005, \; s_0 = 0.65, \quad \delta = 0.000002,$$

$$\delta_0 = 0.001, \quad A = 10, \quad d = 0.0004, \; s = 1, \quad Q_0 = 20, \quad s_1 = 10, \quad L = 5000000.$$

The equilibrium values of $\hat{Y}$, $\hat{N}$ and $\hat{B}$ are found as,

$$\hat{Y} = 1466.742, \quad \hat{N} = 23166.57 \text{ and } \hat{B} = 1986910.242.$$

Simulation is performed for different initial starting points,
In 1, $Y_0 = 3610, \quad N_0 = 5092, \quad B_0 = 2180$.
In 2, $Y_0 = 210, \quad N_0 = 24692, \quad B_0 = 5180$.
In 3, $Y_0 = 4222, \quad N_0 = 24500, \quad B_0 = 1000$.
In 4, $Y_0 = 5000, \quad N_0 = 15000, \quad B_0 = 90$.
In Fig. 3.2, we have plotted the infective population against the susceptible population. From the solution curves, we conclude that the system is globally stable about the endemic equilibrium point $(\hat{Y}, \hat{N}, \hat{B})$ provided we start away from disease free equilibrium. Also in Figs. 3.3-3.8, the variation of the infective population is shown for different values of s, s_1, δ, L and A. It is concluded that with the increase of these parameters, the infective population increases. This implies that with increase in the cumulative discharges into the environment, the spread of bacterial dependent infectious diseases increase.

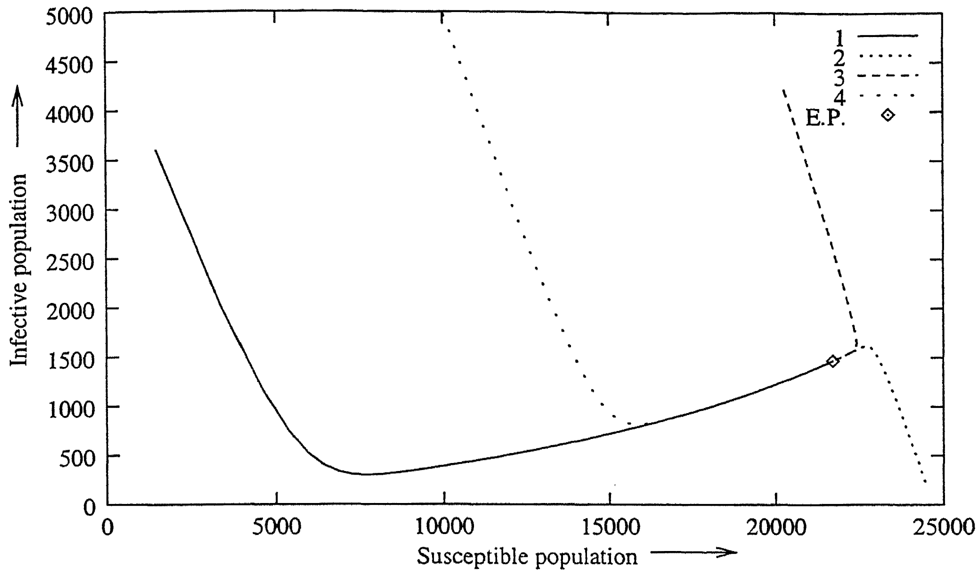


Figure 3.2: Variation of infective population with susceptible population.

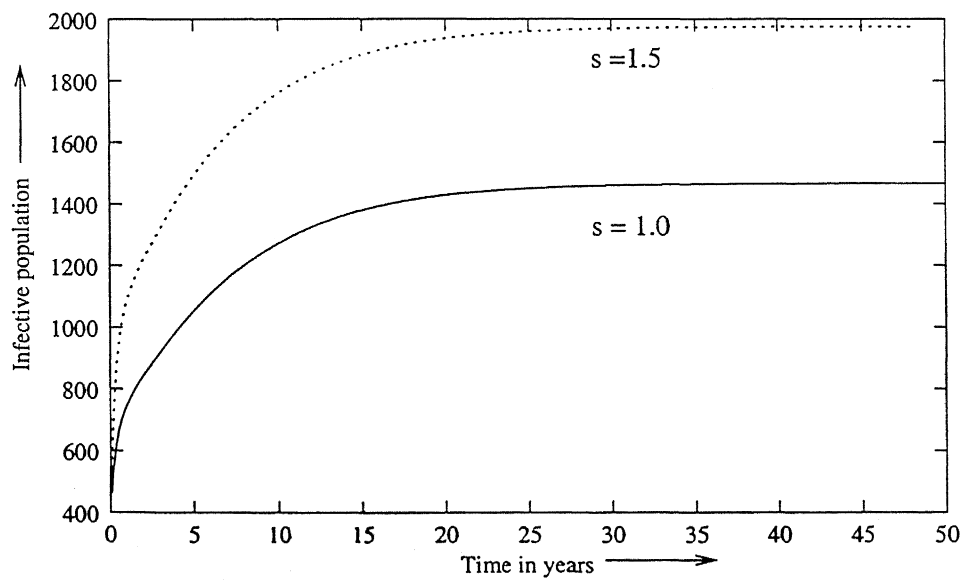


Figure 3.3: Variation of infective population with time for different intrinsic growth rate of bacteria population.

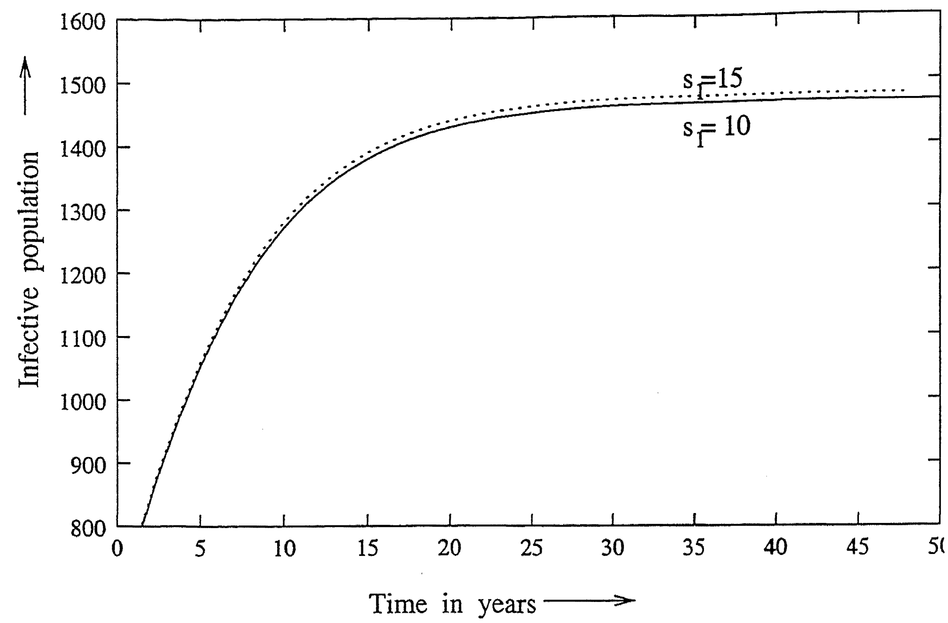


Figure 3.4: Variation of infective population with time for different growth rate of bacteria due to infective human population.

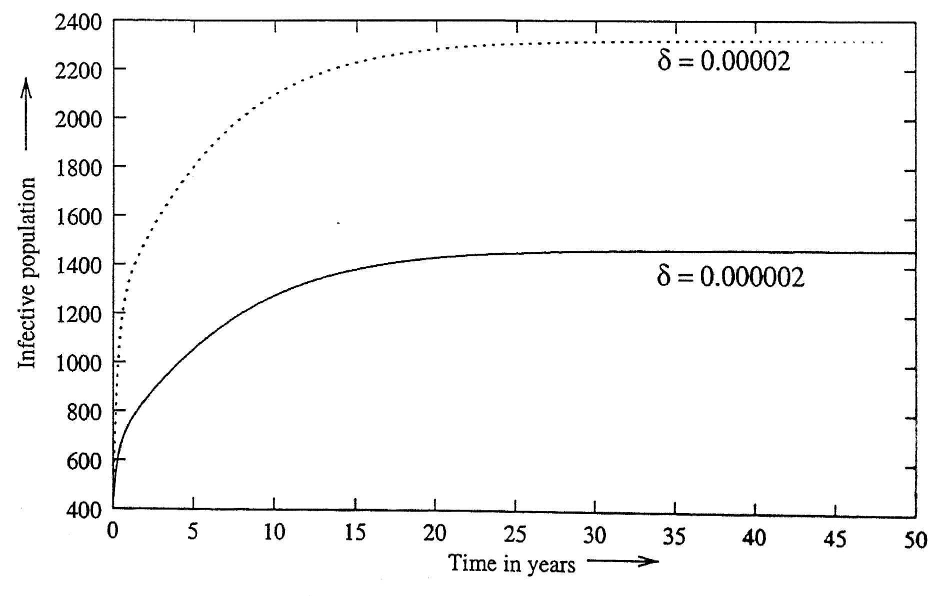


Figure 3.5: Variation of infective population with time for different growth rate of bacteria corresponding to environmental discharges.

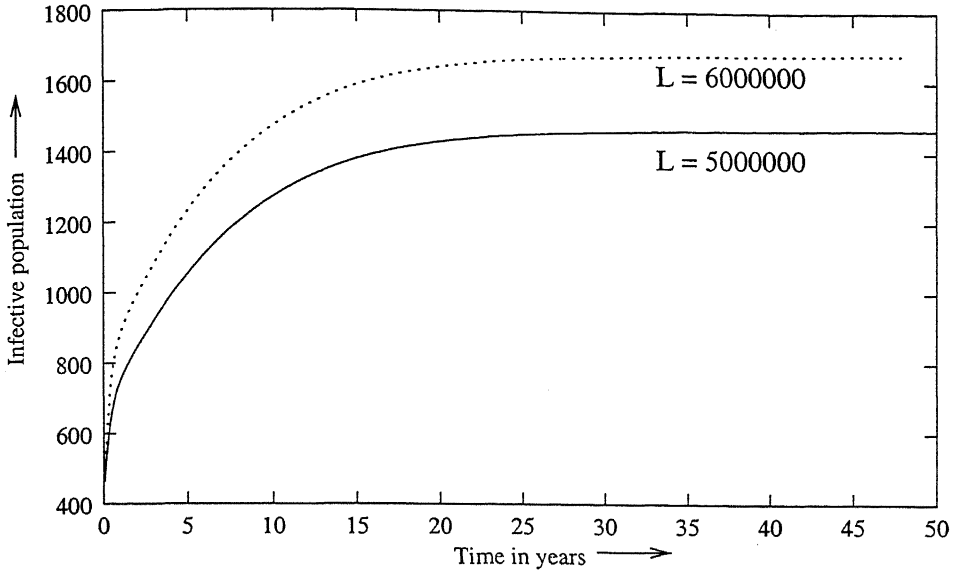


Figure 3.6: Variation of infective population with time for different carrying capacity of bacteria population.

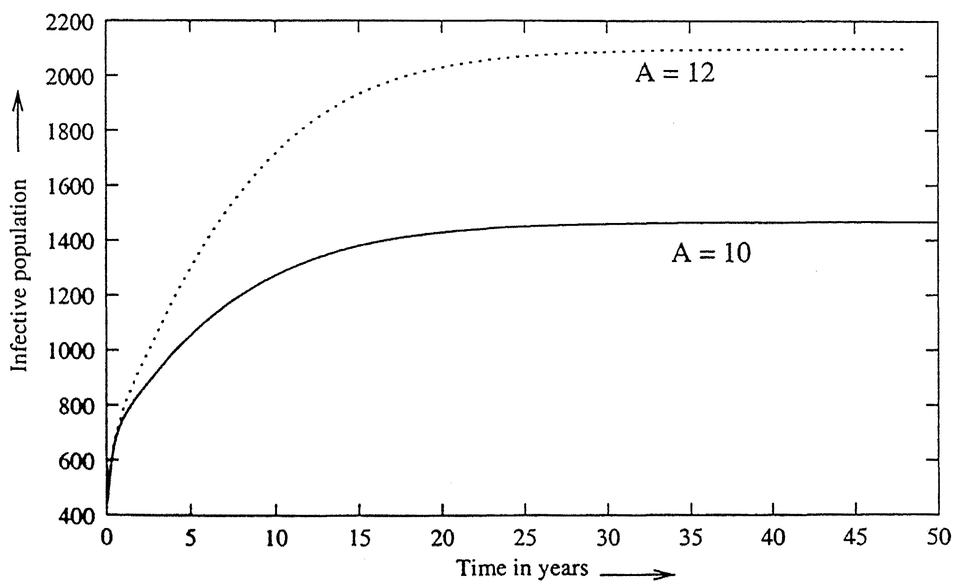


Figure 3.7: Variation of infective population with time for different rate of immigration.

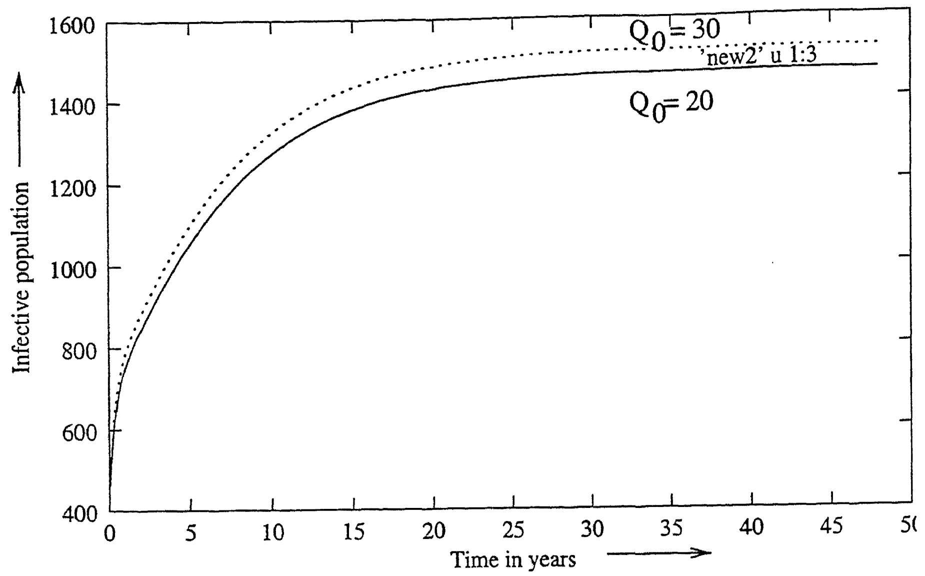


Figure 3.8: Variation of infective population with time for different rate of cumulative environmental discharges

3.2.2 Case II: Q is a Variable

In this case, let us consider the original model (3.1) with $Q(N) = Q_0 + lN$.

$$\begin{aligned}
\dot{Y} &= (\beta Y + \lambda B)(N - Y) - (\nu + \alpha + d)Y, \\
\dot{N} &= A - dN - \alpha Y, \\
\dot{B} &= s\left(B - \frac{B^2}{L}\right) + s_1 Y - s_0 B + \delta\, BE, \\
\dot{E} &= Q_0 + lN - \delta_0 E.
\end{aligned} \tag{3.11}$$

The result of an equilibrium analysis is stated in the following theorem.

THEOREM 3.3 There exist two equilibria, namely (i)$\hat{E}_1\left(0, \frac{A}{d}, 0, \frac{Q_0}{\delta_0 + \frac{lA}{d}}\right)$ *and (ii)*$\hat{E}_2(\hat{Y}, \hat{N}, \hat{B}, \hat{E})$. *The second equilibrium exists if*

$$\frac{L}{s}\{s - s_0 + \delta\frac{Q_0}{\delta_0} + \frac{\delta l A}{\delta_0 d}\} < \frac{s_1 \delta_0 d}{\delta l \alpha}.$$

Proof: We prove the existence of the second equilibrium E_2 as follows. Setting the right hand side of the above system to zero, we get

$$N = \frac{A - \alpha Y}{d}, \quad E = \frac{Q_0 + l\,N}{\delta_0}, \tag{3.12}$$

$$\beta\left(1+\frac{\alpha}{d}\right)Y^2-\left\{\frac{\beta A}{d}-(\nu+\alpha+d)-\lambda(1+\frac{\alpha}{d})B\right\}Y-\frac{\lambda A}{d}B=0, \tag{3.13}$$

$$\text{and } Y=\frac{\frac{s}{L}B^2-\{s-s_0+\delta\frac{Q_0}{\delta_0}+\frac{\delta\ lA}{\delta_0 d}\}B}{s_1-\frac{\delta l\alpha}{\delta_0 d}B}. \tag{3.14}$$

Here (3.13) is the same as (3.4). From (3.14), we get,

(i) when $Y=0$, $B=0$ or $B=\frac{L}{s}\left\{s-s_0+\delta\frac{Q_0}{\delta_0}+\frac{\delta lA}{\delta_0 d}\right\}=B_1>0$ (say),

(ii) $\left(\frac{dY}{dB}\right)_{(0,0)}=-\left\{\frac{s-s_0+\delta\frac{Q_0}{\delta_0}+\frac{\delta lA}{\delta_0 d}}{s_1}\right\}<0$

and

$$\left(\frac{dY}{dB}\right)_{(B_1,0)}=\left\{\frac{s-s_0+\delta\frac{Q_0}{\delta_0}+\frac{\delta lA}{\delta_0 d}}{s_1-\frac{\delta l\alpha}{\delta_0 d}B_1}\right\}>0 \text{ if } B_1<\frac{s_1\delta_0 d}{\delta l\alpha}, \tag{3.15}$$

(iii) $\left(\frac{dY}{dB}\right)\to\infty$ when $B=\frac{s_1\delta_0 d}{\delta l\alpha}$.

As before plotting (3.14) and (3.15), we will get a positive intersecting point $(\hat{B},\hat{Y})$ under condition (3.15). Then $\hat{E}$ and $\hat{N}$ can be calculated from (3.12) as $\hat{Y}<\frac{A}{\alpha}$.

3.2.2.1 Stability Analysis

Now we present the stability analysis of these equilibria. The local stability results of these equilibria are stated in the following theorem.

THEOREM 3.4 The first equilibrium $\hat{E}_1$ is always unstable, and the second equilibrium $\hat{E}_2$ is stable provided $a_3(a_2a_1-a_0a_3)-a_1^2>0$, where a_0, a_1, a_2 and a_3 are given explicitly in the proof of the theorem.

Proof: To study the stability analysis let us consider the variational matrix M corresponding to the system (3.11),

$$M=\begin{pmatrix}\beta N-2\beta Y-\lambda B-(\nu+\alpha+d) & \beta Y+\lambda B & \lambda(N-Y) & 0\\ -\alpha & -d & 0 & 0\\ s_1 & 0 & s-s_0+\delta E-\frac{2Bs}{L} & \delta B\\ 0 & l & 0 & -\delta_0\end{pmatrix}.$$

The variational matrix M_1 corresponding to the system of equation (3.11) at equilibrium point $\hat{E}_1\left(0, \frac{A}{d}, 0, \frac{Q_0+\frac{lA}{d}}{\delta_0}\right)$ is given by

$$M_1 = \begin{pmatrix} \frac{\beta A}{d} - (\nu+\alpha+d) & 0 & \frac{\lambda A}{d} & 0 \\ -\alpha & -d & 0 & 0 \\ s_1 & 0 & s - s_0 + \delta\frac{Q_0+\frac{lA}{d}}{\delta_0} & 0 \\ 0 & l & 0 & -\delta_0 \end{pmatrix}.$$

Here two characteristic roots are $-d$ and $-\delta_0$ and the other roots are given by the following quadratic equation,

$$\psi^2 - \left\{\frac{\beta A}{d} - (\nu+\alpha+d) + s - s_0 + \delta\frac{Q_0+\frac{lA}{d}}{\delta_0}\right\}\psi$$
$$+\left\{\beta\frac{A}{d} - (\nu+\alpha+d)\right\}\left\{s - s_0 + \delta\frac{Q_0+\frac{lA}{d}}{\delta_0}\right\} - \lambda s_1\frac{A}{d} = 0.$$

By the Routh-Hurwitz criteria this equilibrium is unstable because in the above quadratic, the coefficient of ψ and the constant term are not positive simultaneously.

The variational matrix M_2 corresponding to equilibrium point $\hat{E}_2(\hat{Y}, \hat{N}, \hat{B}, \hat{E})$ is given by

$$M_2 = \begin{pmatrix} -(\beta\hat{Y} + \frac{\lambda\hat{N}\hat{B}}{\hat{Y}}) & \beta\hat{Y} + \lambda\hat{B} & \lambda(\hat{N}-\hat{Y}) & 0 \\ -\alpha & -d & 0 & 0 \\ s_1 & 0 & -\left(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right) & \delta\hat{B} \\ 0 & l & 0 & -\delta_0 \end{pmatrix}.$$

The characteristic polynomial is given by

$$\psi^4 + a_3\,\psi^3 + a_2\psi^2 + a_1\psi + a_0 \;=\; 0,$$

where

$$\begin{aligned}
a_3 &= \left(\beta\hat{Y} + \frac{\lambda\hat{N}\hat{B}}{\hat{Y}}\right) + d + \frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}} + \delta_0, \\
a_2 &= \left(\beta\hat{Y} + \frac{\lambda\hat{N}\hat{B}}{\hat{Y}}\right)\left(d + \frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}} + \delta_0\right) + d\left(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}} + \delta_0\right) + \left(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right)\delta_0 \\
&\quad +\alpha(\beta\hat{Y} + \lambda\hat{B}) - \lambda(\hat{N} - \hat{Y})s_1, \\
a_1 &= \left(\beta\hat{Y} + \frac{\lambda\hat{N}\hat{B}}{\hat{Y}}\right)d\left(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}} + \delta_0\right) + \left(\beta\hat{Y} + \frac{\lambda\hat{N}\hat{B}}{\hat{Y}} + d\right)\left(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right)\delta_0 \\
&\quad +\alpha(\beta\hat{Y} + \lambda\hat{B})\left(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}} + \delta_0\right) - \lambda(\hat{N} - \hat{Y})s_1(d+\delta_0), \\
a_0 &= \left(\beta\hat{Y} + \frac{\lambda\hat{N}\hat{B}}{\hat{Y}}\right)d\left(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right)\delta_0 + \alpha(\beta\hat{Y} + \lambda\hat{B})\delta_0\left(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right) \\
&\quad -\lambda(\hat{N} - \hat{Y})(ds_1\delta_0 - \alpha l\delta\hat{B}) > 0.
\end{aligned}$$

By Murata (1977), conditions for local stability of the system are

$$a_3 > 0, \quad \begin{vmatrix} a_3 & a_1 \\ 1 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_3 & a_1 & 0 \\ 1 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0, \quad \begin{vmatrix} a_3 & a_1 & 0 & 0 \\ 1 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & 1 & a_2 & a_0 \end{vmatrix} > 0.$$

The first two inequalities are obvious. If the third is satisfied then so is the fourth one as $a_0 > 0$. Thus the equilibrium $\hat{E}_2$ is locally asymptotically stable under the condition mentioned in the theorem.

Nonlinear Analysis and Simulation:

The global stability of $\hat{E}_2$ can be speculated as before. To show this, the system (3.11) is integrated using the fourth order Runge-Kutta Method and using same parameters as mentioned in the previous subsection, with $Q_0 = Q_a$ and an additional parameter $l = 0.000005$, which satisfy the local stability conditions. The equilibrium values of $\hat{Y}$, $\hat{N}$, $\hat{B}$ and $\hat{E}$ have been found as

$$\hat{Y} = 1467.388, \quad \hat{N} = 23165.766, \quad \hat{B} = 1988063.370, \quad \hat{E} = 20115.828.$$

In this case also simulation is performed for different initial positions 1, 2, 3 and 4, shown in Fig. 3.9, where the infective population is plotted against the susceptible population. From the solution curves, it appears plausible that system is globally stable about the endemic equilibrium point $(\hat{Y},\ \hat{N},\ \hat{B},\ \hat{E})$ under the local stability condition provided we start away from the other equilibrium point. Also in Figs. 3.10-3.15 the variation in the infective population is shown for different values of s, s_1, δ, L, l and A. It is concluded that with the increase of these parameters , the infective population increases showing that the spread of bacterial infectious disease increases with bad environmental conditions as well as with immigration.

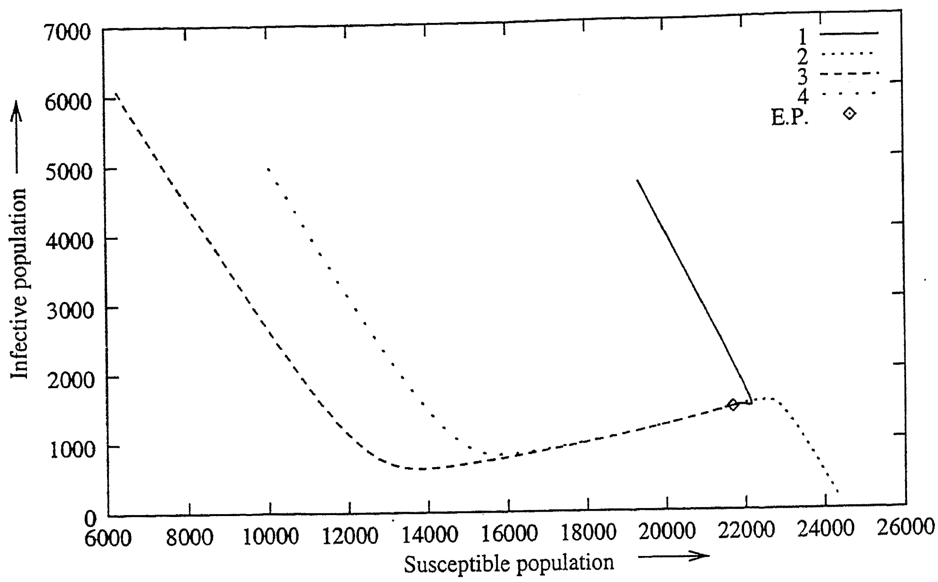


Figure 3.9: Variation of infective population with susceptible population.

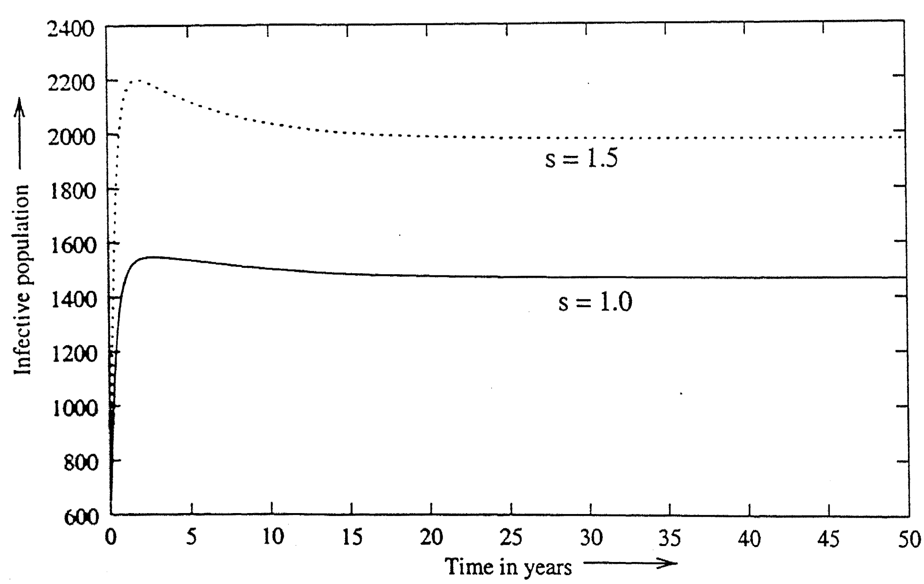


Figure 3.10: Variation of infective population with time for different intrinsic growth rate of bacteria population.

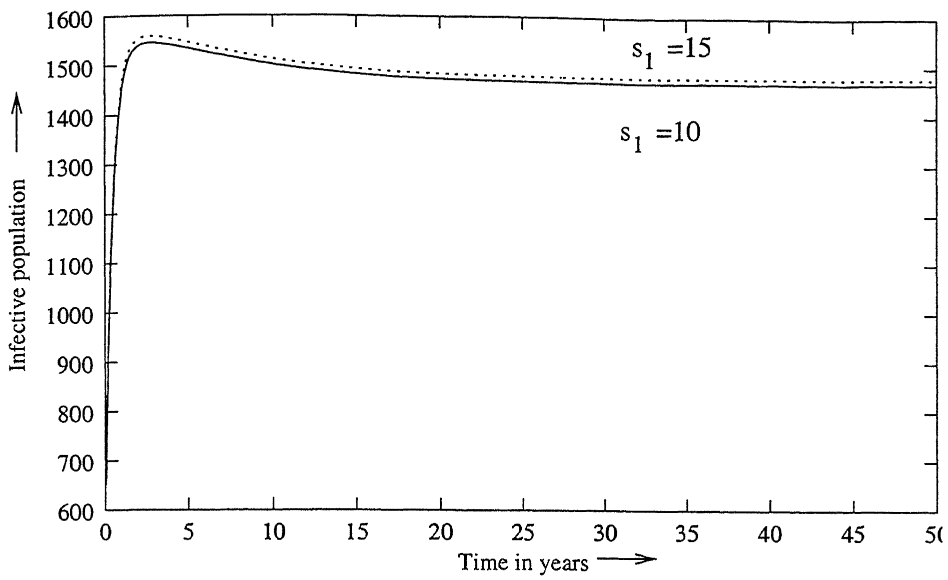


Figure 3.11: Variation of infective population with time for different growth rate of bacteria population due to infective human population.

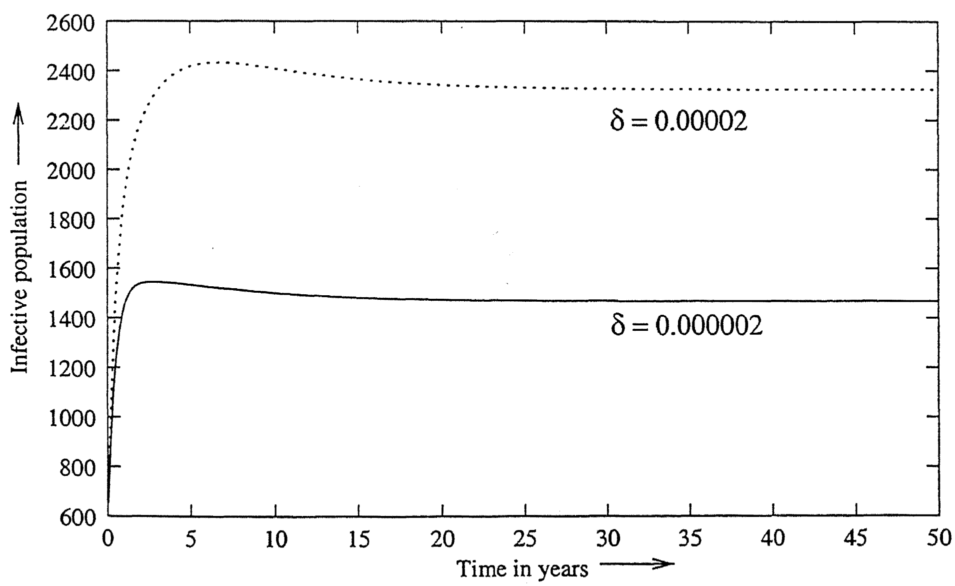


Figure 3.12: Variation of infective population with time for different growth rate of bacteria population corresponding to environmental discharges.

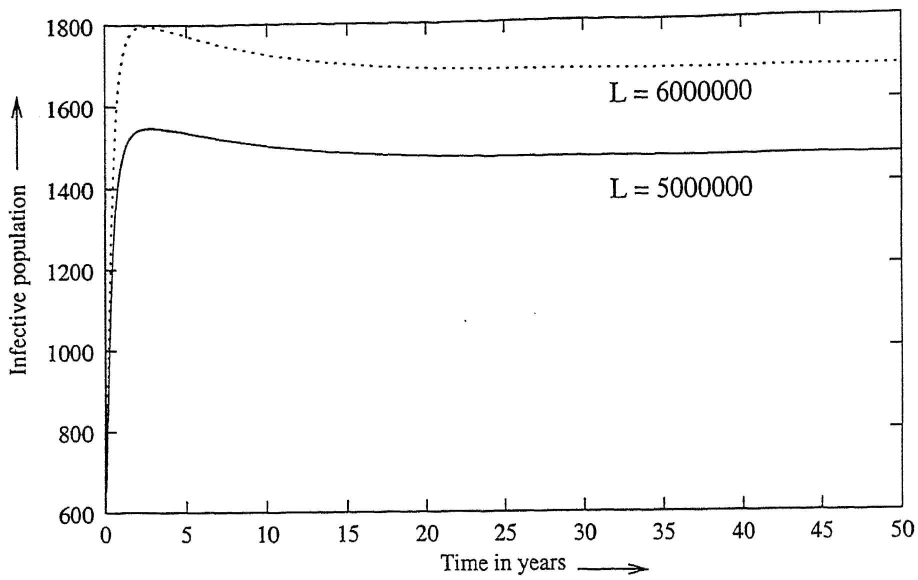


Figure 3.13: Variation of infective population with time for different carrying capacity of bacteria population.

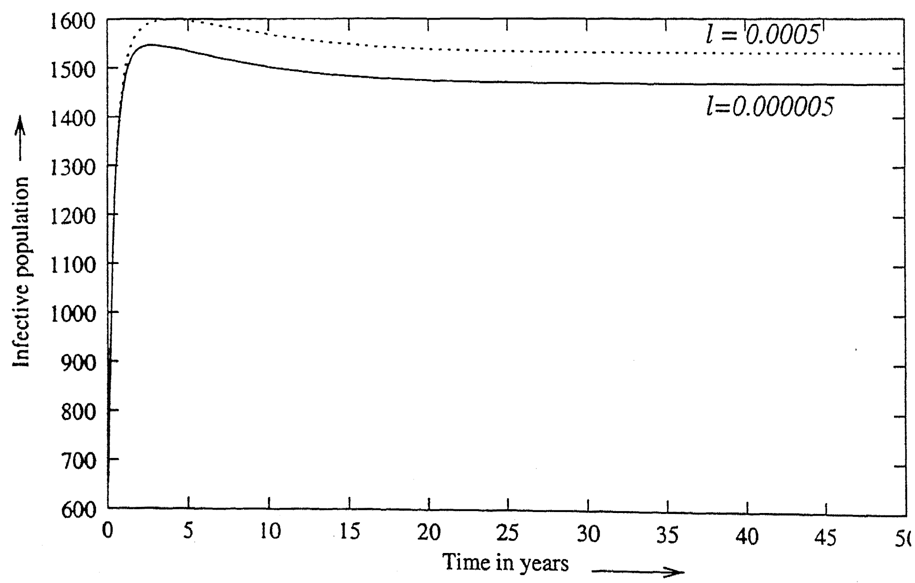


Figure 3.14: Variation of infective population with time for different l.

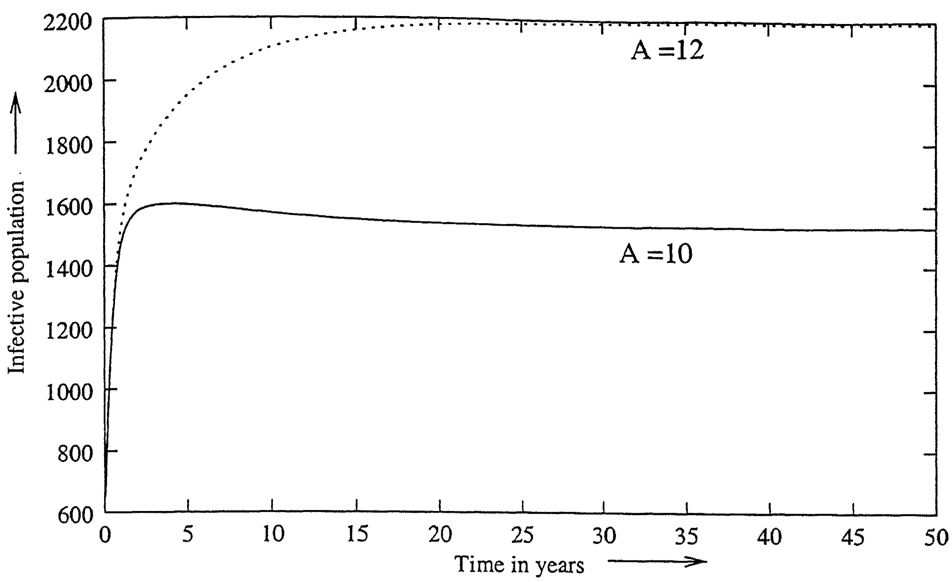


Figure 3.15: Variation of infective population with time for different rate of immigration of human population.

3.3 SIS Model with Logistic Growth

In this case, we consider an SIS model in which logistic growth of human population is assumed. Here both the birth and death rates are density dependent. The birth rate decreases and the death rate increases as the population size increases towards its carrying capacity (Gao and Hethcote 1992). The mathematical model is given by the following set of equations,

$$\begin{aligned}
\dot{X} &= \left\{b - a\,r\frac{N}{K}\right\} N - \left\{d + (1-a)\frac{rN}{K}\right\} X - (\beta Y + \lambda B)X + \nu Y, \qquad 0 \le a \le 1, \\
\dot{Y} &= (\beta Y + \lambda B)X - \left\{\nu + \alpha + d + (1-a)\frac{rN}{K}\right\} Y, \\
\dot{N} &= r\left(1 - \frac{N}{K}\right) N - \alpha Y, \\
\dot{B} &= sB\left(1 - \frac{B}{L}\right) + s_1 Y - s_0 B + \delta B E, \\
\dot{E} &= Q(N) - \delta_0 E = Q_0 + lN - \delta_0 E.
\end{aligned} \tag{3.16}$$

Here b and d are the natural birth and death rates; $r = b - d > 0$ is the growth rate constant; K is the carrying capacity of the environment corresponding to the human population. All other parameters are as defined in the previous subsection. For $0 < a < 1$,

the birth rate decreases and the death rate increases as N increases to its carrying capacity K. When $a = 1$, the model could be called simply a logistic birth model as all of the restricted growth is due to a decreasing birth rate and the death rate is constant. Similarly, when $a = 0$, it could be called a logistic death model as all of the restricted growth is due to an increasing death rate and the birth rate is constant. We see that the region of attraction T' corresponding to (3.16) is

$$T' = \left\{(Y, N, B, E) : 0 \leq Y < \frac{rK}{4\alpha},\ 0 < N \leq K,\ 0 \leq B \leq B_{\max},\ 0 \leq E \leq \frac{Q(K)}{\delta_0}\right\},$$

and the model is well-posed in the region T', where $B_{\max}$ is given by

$$B_{\max} = \frac{L}{2s}\left[\left\{s - s_0 + \delta\frac{Q(K)}{\delta_0}\right\} + \sqrt{\left\{s - s_0 + \delta\frac{Q(K)}{\delta_0}\right\}^2 + \frac{4ss_1K}{L}}\,\right].$$

3.3.1 Case I: Q is a Constant Q_a

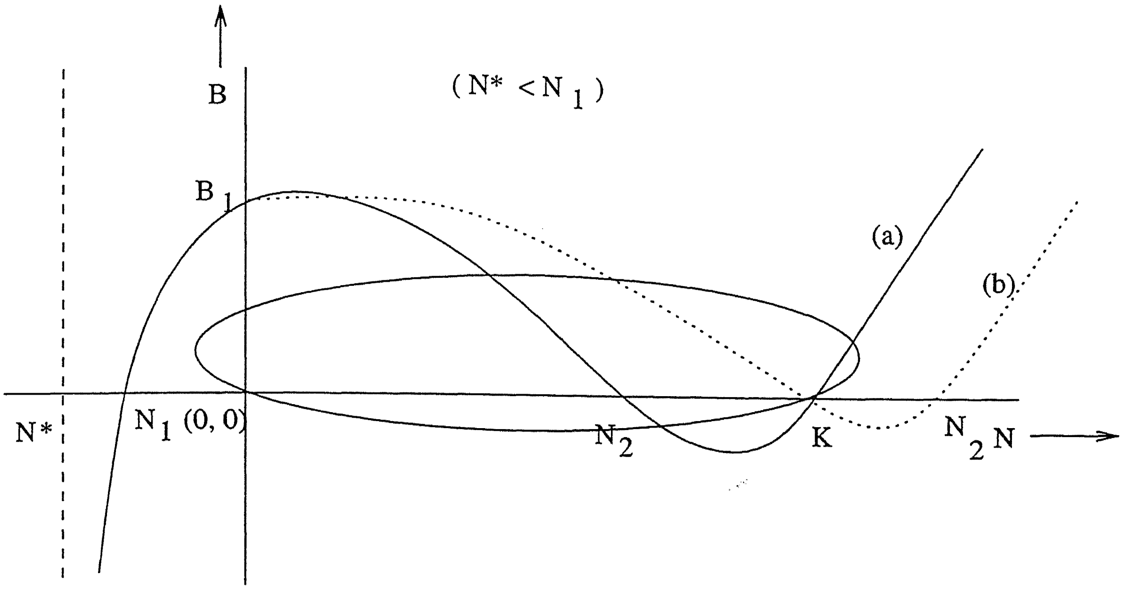
Since $X + Y = N$, it is sufficient to consider the following equivalent system of (3.16),

$$\begin{aligned}
\dot{Y} &= (\beta Y + \lambda B)(N - Y) - \left\{\nu + \alpha + d + (1-a)\frac{rN}{K}\right\}Y, \\
\dot{N} &= r\left(1 - \frac{N}{K}\right)N - \alpha Y, \\
\dot{B} &= sB\left(1 - \frac{B}{L}\right) + s_1Y - s_0B + \delta BE, \\
\dot{E} &= Q_a - \delta_0 E.
\end{aligned} \tag{3.17}$$

Putting the asymptotic value of E i.e. $E_m = \frac{Q_a}{\delta_0}$ in the above system of equations we get the following subsystem:

$$\begin{aligned}
\dot{Y} &= (\beta Y + \lambda B)(N - Y) - \{\nu + \alpha + d + (1-a)\frac{rN}{K}\}Y, \\
\dot{N} &= r(1 - \frac{N}{K})N - \alpha Y, \\
\dot{B} &= sB\,(1 - \frac{B}{L}) + s_1Y - s_0B + \delta B\frac{Q_a}{\delta_0}.
\end{aligned} \tag{3.18}$$

The result of equilibrium analysis is stated in the following theorem.



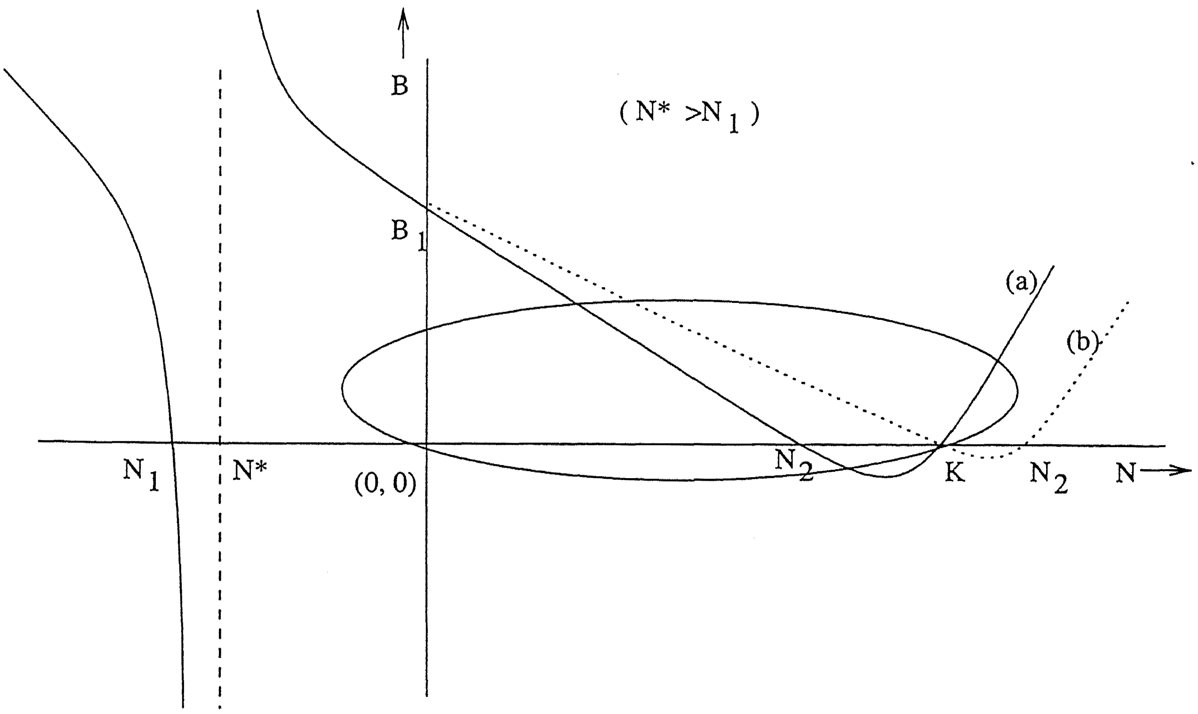


Figure 3.16: Existence of equilibrium point when $\alpha > r$.

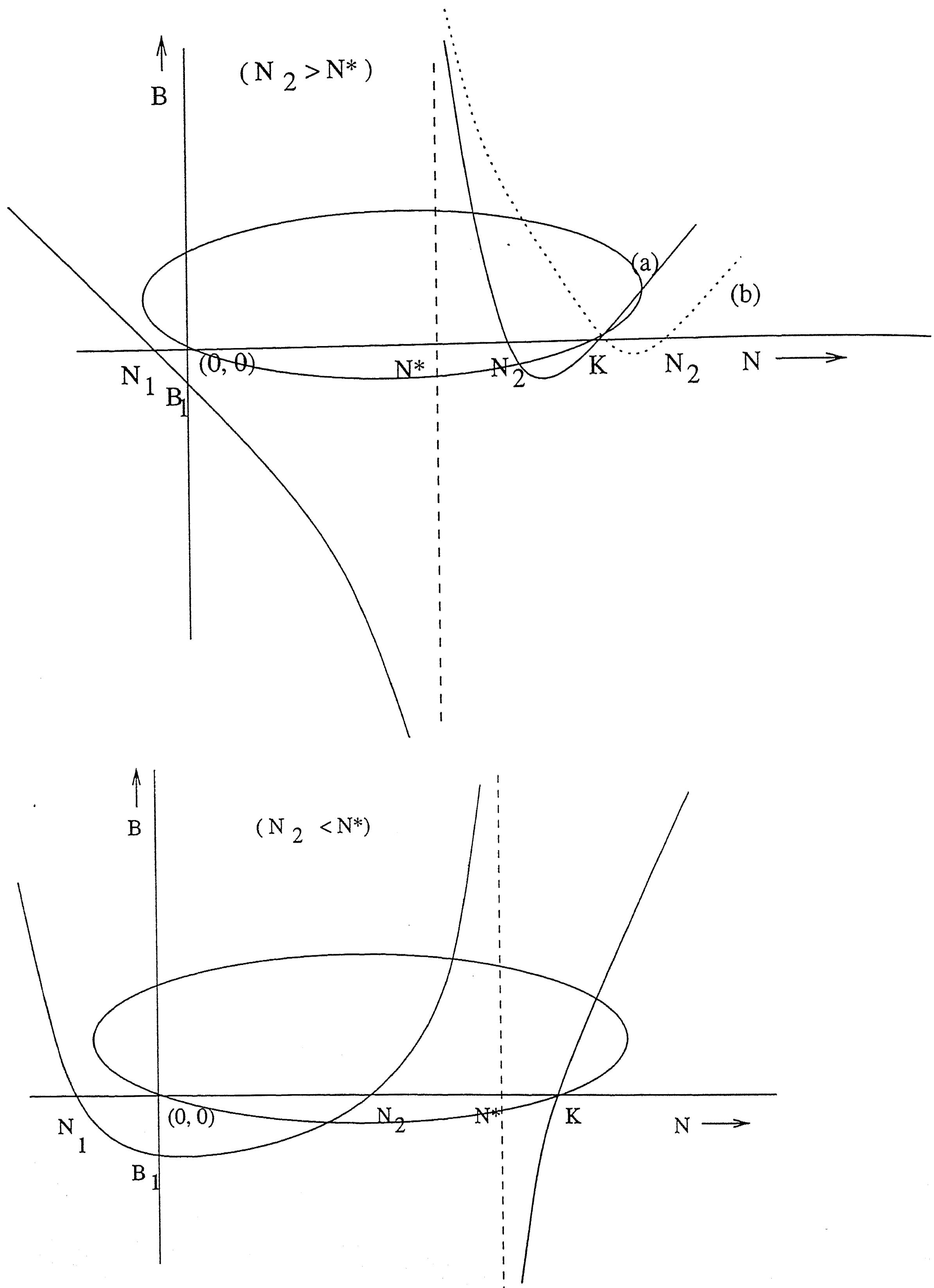


Figure 3.17: Existence of equilibrium point when $\alpha < r$.

THEOREM 3.5 There exist the following four equilibria, namely (i)$P_1(0,0,0)$, *(ii)*$P_2(0,0,B^*)$, *where* $B^* = \frac{L}{s}\{s - s_0 + \delta\frac{Q_a}{\delta_0}\} > 0$, *(iii)* $P_3(0,K,0)$ *and (iv)* $P_4(\hat{Y},\hat{N},\hat{B})$, *for* $\alpha > r$, *this exists if*

$$\nu + \alpha + d > \frac{(\alpha - r)}{r}\lambda B^*.$$

Proof: The existence of the first three equilibria is obvious. We will show the existence of the fourth equilibrium point P_4 by the isocline method. Setting the right hand sides of (3.18) to zero, we get the following for $N \neq 0$ and $N \neq K$.

$$Y = \frac{r}{\alpha}N\left(1 - \frac{N}{K}\right), \tag{3.19}$$

$$\frac{s}{L}B^2 - \left(s - s_0 + \delta\frac{Q_a}{\delta_0}\right)B - \frac{s_1 r}{\alpha}(1 - \frac{N}{K})N = 0, \tag{3.20}$$

$$\left[\frac{k_1}{K}N - \left\{\beta\frac{r}{\alpha}N(1 - \frac{N}{K}) + \lambda B + \nu + \alpha + d\right\}\right]\frac{r}{\alpha}(1 - \frac{N}{K}) + \lambda B = 0, \tag{3.21}$$

where $k_1 = \beta K - (1 - a)r$.

Also (3.21) can be written as

$$B = \frac{\frac{\beta r^2}{\alpha^2 K^2}(N - K)\left\{N^2 + \frac{\alpha}{\beta r}\left(k_1 - \frac{\beta r}{\alpha}\frac{K}{}\right)N - \frac{\alpha K}{\beta r}(\nu + \alpha + d)\right\}}{\frac{\lambda r}{\alpha K}\left\{N - \left(1 - \frac{\alpha}{r}\right)K\right\}},$$

or

$$B = \frac{\beta r}{\lambda\alpha K}\frac{(N - K)(N - N_1)(N - N_2)}{(N - N^*)}, \tag{3.22}$$

where N_1 is the negative and N_2 is the positive root corresponding to the quadratic in the numerator of the above equation and $N^* = (1 - \frac{\alpha}{r})K$.

Also when $N = 0$, $B = -\dfrac{(\nu + \alpha + d)}{\lambda(1 - \frac{\alpha}{r})} = B_1$ (say).

Clearly in the N-B plane, (3.20) is an ellipse passing through (0, 0) and $(K, 0)$ with major and minor axes parallel to the coordinate axes and origin at $\left(\frac{K}{2}, \frac{L}{2s}\left(s - s_0 + \delta\frac{Q_a}{\delta_0}\right)\right)$.

We consider the following two cases:

Case I: $\alpha > r$, i.e. N^* is negative

In this case, (a) when $k_1 > \nu + \alpha + d > 0 \Rightarrow 0 < N_2 < K$ and
(b) when k_1 is negative or $0 < k_1 < \nu + \alpha + d \Rightarrow K < N_2 < \infty$.
The graphs of (3.20) and (3.22) are shown in Fig. 3.16. We note that for the existence of positive intersecting point with N less than carrying capacity K, we must have $B_1 > B_2$, where B_2 is given by (3.20) when $N = 0$. We see that $B_2 = B^*$. Thus in this case, the condition for existence of $(\hat{N}, \hat{B})$ is

$$\nu + \alpha + d > \frac{(\alpha - r)}{r}\lambda B^*.$$

Case II: $0 < \alpha < r$, i.e. N^* is positive.
In this case, (a) when $k_1 > \nu + \alpha + d > 0 \Rightarrow 0 < N_2 < K$ and
(b) when k_1 is negative or $0 < k_1 < \nu + \alpha + d \Rightarrow K < N_2 < \infty$.
The graphs of (3.20) and (3.22) are shown in Fig. 3.17. It is noted that the equilibrium value $\hat{N}_1 > \frac{K}{2}$ surely, if (i) $N^* > \frac{K}{2}$ and (ii) $N_2 > \frac{K}{2}$, which give rise to the following sufficient conditions on the parameters,

$$0 < \alpha < \frac{r}{2}, \quad \frac{k_1}{2} - (\nu + \alpha + d) - \frac{\beta r K}{4\alpha} < 0. \tag{3.23}$$

3.3.1.1 Stability Analysis

Now we present the stability analysis of these equilibria. The local stability results of these equilibria are stated in the following theorem.

THEOREM 3.6 The equilibrium P_1 is unstable, the equilibrium P_2 is locally asymptotically stable provided

$$\frac{(\alpha - r)}{r}\lambda B^* > \nu + \alpha + d,$$

otherwise if

$$\frac{(\alpha - r)}{r}\lambda B^* < \nu + \alpha + d$$

it is unstable. The equilibrium P_3 is unstable and the equilibrium point P_4 is locally asymptotically stable if the condition $a_3 > 0$ and $a_1 a_2 - a_3 > 0$ are satisfied, where a_1, a_2 and a_3 are given explicitly in the proof of the theorem.

Proof: The variational matrix M at (Y, N, B) corresponding to the system (3.18), is given by

$$M = \begin{pmatrix} \beta(N-2Y) - \lambda B - \{\nu+\alpha+d+(1-a)\frac{rN}{K}\} & \beta Y + \lambda B & \lambda(N-Y) \\ -\alpha & (r - 2r\frac{N}{K}) & 0 \\ s_1 & 0 & s - s_0 + \delta\frac{Q_a}{\delta_0} - 2\frac{s}{L}B \end{pmatrix},$$

The variational matrix M_1 at equilibrium point $P_1(0,0,0)$ is

$$M_1 = \begin{pmatrix} -(\nu+\alpha+d) & 0 & 0 \\ -\alpha & r & 0 \\ s_1 & 0 & s - s_0 + \delta\frac{Q_a}{\delta_0} \end{pmatrix}.$$

Clearly one eigenvalue of the above matrix is positive, this equilibrium is unstable.

The variational matrix M_2 at equilibrium point P_2 is

$$M_2 = \begin{pmatrix} -\{\lambda B^* + \nu + \alpha + d\} & \lambda B^* & 0 \\ -\alpha & r & 0 \\ s_1 & 0 & -(s - s_0 + \delta\frac{Q_a}{\delta_0}) \end{pmatrix}.$$

Clearly one eigenvalue is negative and the other eigenvalues are given by the following quadratic

$$\psi^2 + \{\lambda B^* + \nu + \alpha + d - r\}\psi + \alpha\lambda B^* - r(\lambda B^* + \nu + \alpha + d) = 0.$$

So this equilibrium is stable provided $\frac{(\alpha - r)}{r}\lambda B^* > \nu + \alpha + d$. Clearly for $\alpha < r$, this equilibrium is unstable.

The variational matrix M_3 at the equilibrium point P_3 is

$$M_3 = \begin{pmatrix} \beta K - \{\nu+\alpha+d+(1-a)r\} & 0 & \lambda K \\ -\alpha & -r & 0 \\ s_1 & 0 & s - s_0 + \delta\frac{Q_a}{\delta_0} \end{pmatrix}.$$

The characteristic polynomial corresponding to the above matrix is

$$\begin{aligned}(r+\psi)\Big[\psi^2 - \Big\{s - s_0 + \delta\frac{Q_a}{\delta_0} + \beta K - (\nu+\alpha+d+(1-a)r)\Big\}\psi \\ + \Big(s - s_0 + \delta\frac{Q_a}{\delta_0}\Big)\{\beta K - (\nu+\alpha+d+(1-a)r) - \lambda K s_1\Big] = 0\end{aligned} \tag{3.24}$$

So one root is negative and the other roots have negative real part if

$$\beta K - (1-a)r - (\nu+\alpha+d) + s - s_0 + \delta\frac{Q_a}{\delta_0} < 0, \tag{3.25}$$

and also,

$$(s - s_0 + \delta\frac{Q_a}{\delta_0})[\beta K - (1-a)r - (\nu + \alpha + d)] \; > \; \lambda K s_1. \tag{3.26}$$

Clearly both conditions are not satisfied simultaneously, so this equilibrium is unstable.

Now the variational matrix M_4 at equilibrium point P_4 is given by

$$M_4 = \begin{pmatrix} -(\beta\hat{Y} + \frac{\lambda\hat{B}\hat{N}}{\hat{Y}}) & \beta\hat{Y} + \lambda\hat{B} - (1-a)\frac{r\hat{Y}}{K} & \lambda(\hat{N} - \hat{Y}) \\ -\alpha & r - \frac{2r\hat{N}}{K} & 0 \\ s_1 & 0 & -(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}) \end{pmatrix}$$

The characteristic polynomial corresponding to the above matrix is

$$\psi^3 + a_1\psi^2 + a_2\psi + a_3 = 0,$$

where

$$\begin{aligned} a_1 &= \beta\hat{Y} + \frac{\lambda\hat{B}\hat{N}}{\hat{Y}} + \frac{r}{K}(2\hat{N} - K) + \frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}, \\ a_2 &= (\beta\hat{Y} + \frac{\lambda\hat{B}\hat{N}}{\hat{Y}})\left\{\frac{r}{K}(2\hat{N} - K) + \frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right\} + \frac{r}{K}(2\hat{N} - K)\left\{\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right\} \\ &\quad + \alpha\left\{\beta\hat{Y} + \lambda\hat{B} - (1-a)\frac{r\hat{Y}}{K}\right\} - \lambda s_1(\hat{N} - \hat{Y}), \\ a_3 &= \left(\beta\hat{Y} + \frac{\lambda\hat{B}\hat{N}}{\hat{Y}}\right)\left(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right)\frac{r}{K}(2\hat{N} - K) \\ &+ \alpha\left\{\beta\hat{Y} + \lambda\hat{B} - (1-a)\frac{r\hat{Y}}{K}\right\}\left(\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right) - \frac{r}{K}(2\hat{N} - K)\lambda s_1(\hat{N} - \hat{Y}). \end{aligned}$$

Hence by the Routh-Hurwitz criteria, the system is locally stable if $a_1 > 0$, $a_3 > 0$ and $a_1\,a_2 - a_3 > 0$. However since $a_1 > 0$, the system is stable if $a_3 > 0$ and $a_1\,a_2 - a_3 > 0$ and it is unstable if $a_3 < 0$ or $a_1\,a_2 - a_3 < 0$.

Remark: It is easily seen that the local stability conditions are satisfied if $\hat{N} > \frac{K}{2}$. Hence for numerical purposes, the set of parameters are chosen in such a way that $\hat{N} > \frac{K}{2}$.

Nonlinear Analysis and Simulation:

We first analyze the model (3.18) for $\alpha = 0$. As before it can be shown that the nontrivial equilibrium point of the model (3.18) is globally stable by taking the following Liapunov function,

$$V = \frac{1}{2}(Y - \hat{Y})^2 + p_1(N - \hat{N} - \hat{N}ln\frac{N}{\hat{N}}) + \frac{1}{2}p_2(B - \hat{B})^2,$$

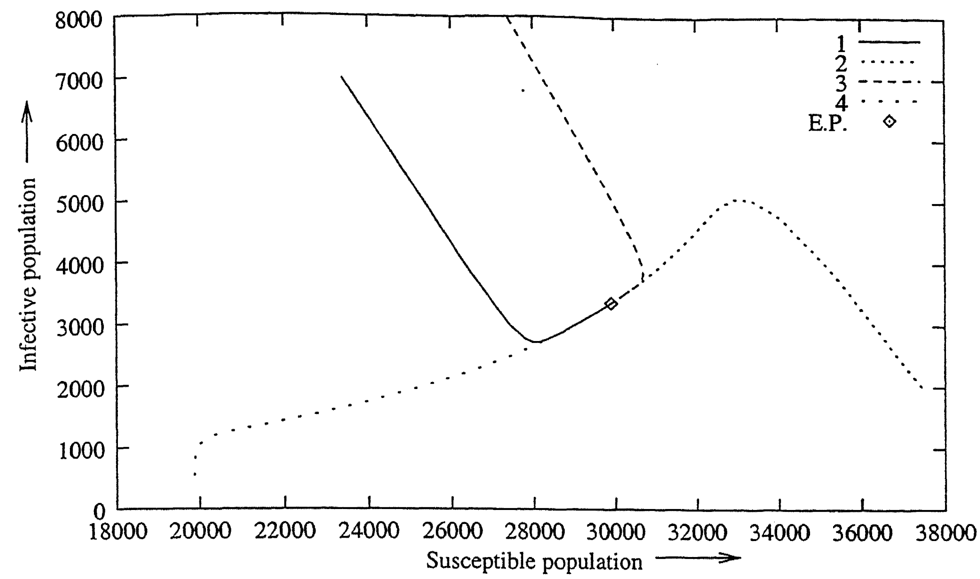


Figure 3.18: Variation of infective population with susceptible population.

where $p_1 > max\left\{\frac{[\beta K-(1-a)r]^2}{\beta r}, \frac{2\lambda LK}{r}\right\}$ and $p_2 = \frac{\lambda(\hat{N}-\hat{Y})}{s_1}$.

As the system (3.18) is bounded by its corresponding system with $\alpha = 0$, which is globally stable, one may speculate that the equilibrium P_4 may be globally stable. The system (3.18) is integrated by considering the following set of parameters, which satisfy the local stability condition.

$$\beta = 0.00000031, \quad \lambda = 0.00000000021, \quad \nu = 0.012, \quad \alpha = 0.0005,$$

$$d = 0.0004, \quad a = 0.3, \quad r = 0.0003, \quad K = 40000, \quad s = 1, \quad L = 5000000,$$

$$s_0 = 0.65, \quad s_1 = 10, \quad \delta = 0.000002, \quad Q_0 = 20, \quad \delta_0 = 0.001.$$

The equilibrium values for this set of parameters are obtained as

$$\hat{Y} = 3358.354, \quad \hat{N} = 33270.632, \quad \hat{B} = 2032611.913.$$

Simulation is performed for different initial positions 1, 2, 3 and 4 as shown in Fig. 3.18. From this figure, it is clear that this equilibrium may be globally stable provided that we start away from the other equilibria. Also in Figs. 3.19-3.23, the variation of infective population is shown for different s, s_1, δ, L and A respectively. It is concluded that with the increase of these parameters, the infective population increases.

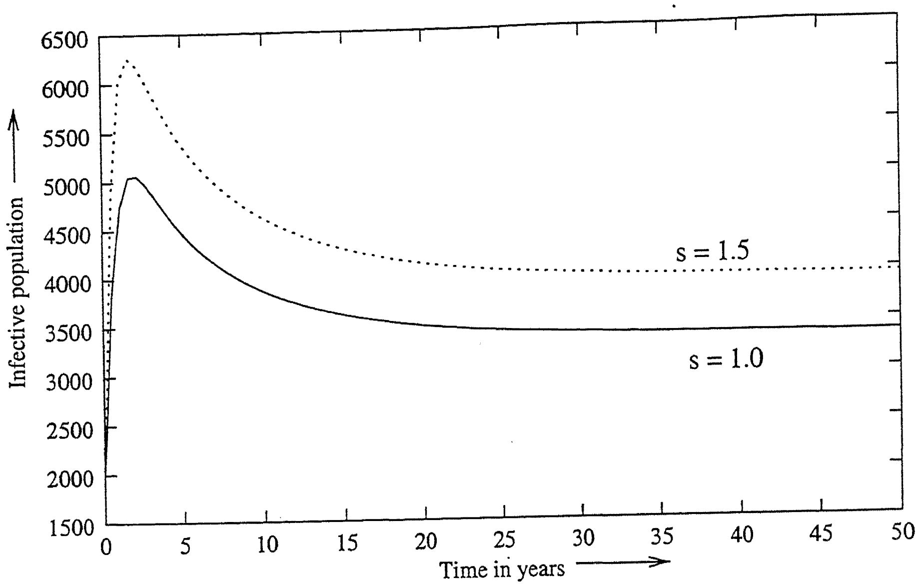


Figure 3.19: Variation of infective population with time for different intrinsic growth rate of bacteria population.

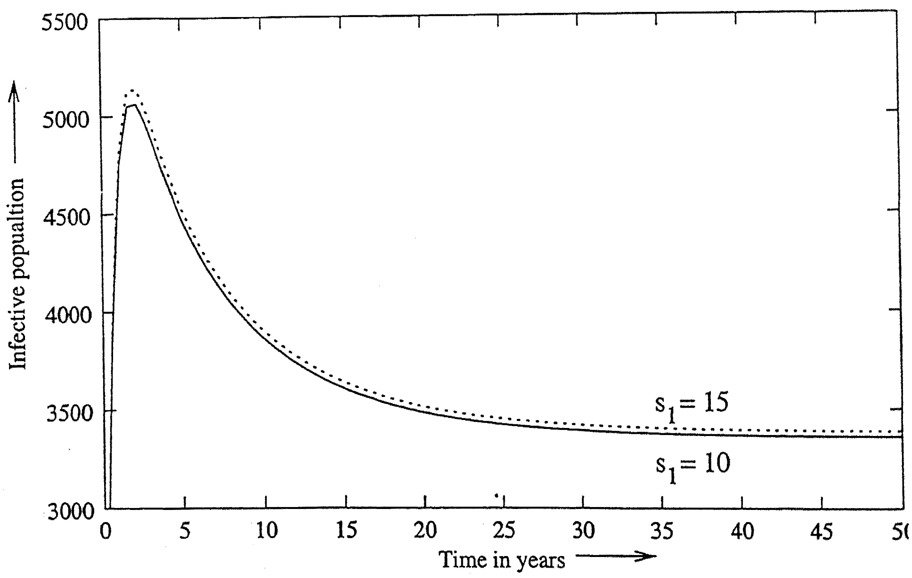


Figure 3.20: Variation of infective population with time for different growth rate of bacteria population due to infective human population.

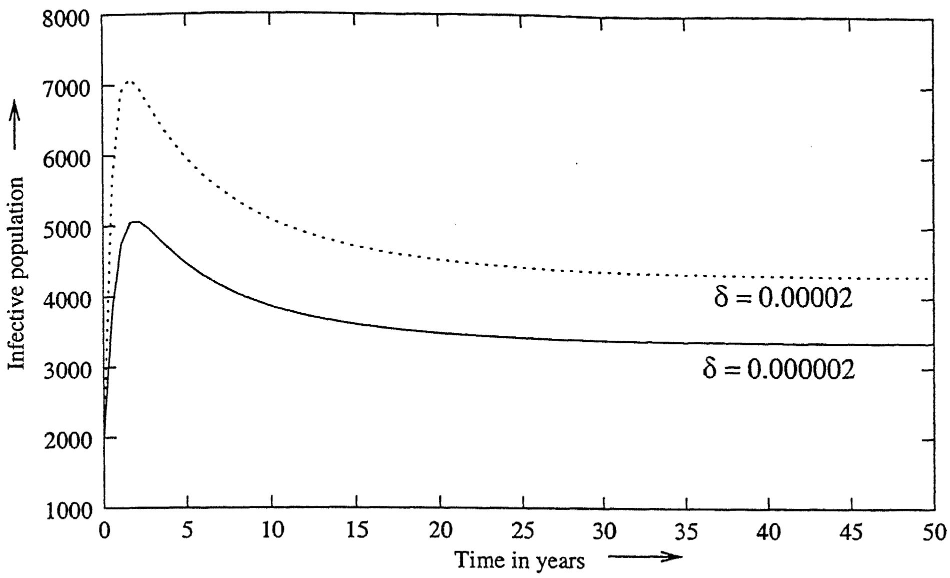


Figure 3.21: Variation of infective population with time for different growth rate of bacteria corresponding to environmental discharges.

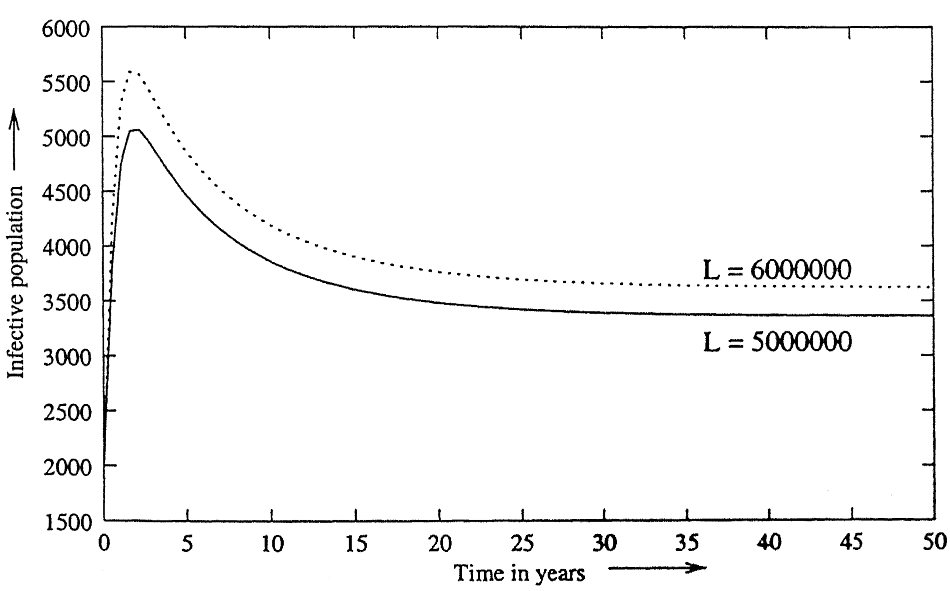


Figure 3.22: Variation of infective population with time for different carrying capacity of bacteria population.

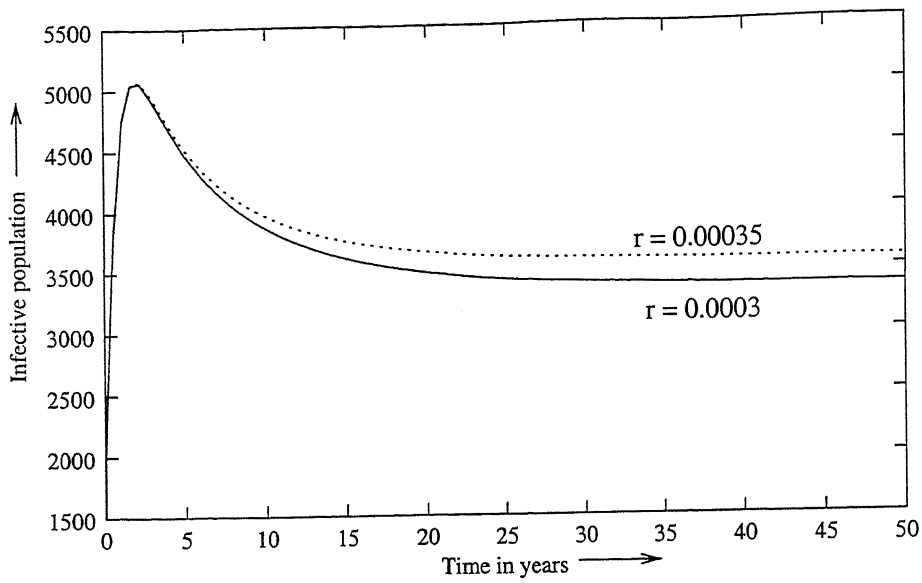


Figure 3.23: Variation of infective population with time for different rate of immigration of human population.

3.3.2 Case II: Q is a Variable

In this case let us consider following system of equations

$$
\begin{aligned}
\dot{Y} &= (\beta Y + \lambda B)(N - Y) - \{\nu + \alpha + d + (1-a)\frac{rN}{K}\}Y, \\
\dot{N} &= r\left(1 - \frac{N}{K}\right)N - \alpha Y, \\
\dot{B} &= sB\left(1 - \frac{B}{L}\right) + s_1 Y - s_0 B + \delta BE, \\
\dot{E} &= Q_0 + lN - \delta_0 E.
\end{aligned}
\tag{3.27}
$$

The result of an equilibrium analysis is stated in the following theorem.

THEOREM 3.7 There exist the following four equilibria, namely (i) $E_1(0,0,0,\frac{Q_0}{\delta_0})$, (ii) $E_2(0,0,B^*,\frac{Q_0}{\delta_0})$, where $B^* = \frac{L}{s}\left\{s - s_0 + \delta\frac{Q_0}{\delta_0}\right\}$, (iii) $E_3(0,K,0,\frac{Q_0+lK}{\delta_0})$ and (iv) $E_4(\hat{Y},\hat{N},\hat{B},\hat{E})$, this exists if

$$\frac{4ss_1r}{\alpha KL} > \left(\frac{\delta l}{\delta_0}\right)^2.$$

For $\alpha > r$, we should have an additional condition as $\nu + \alpha + d > \dfrac{(\alpha - r)}{r}\lambda B^*$.

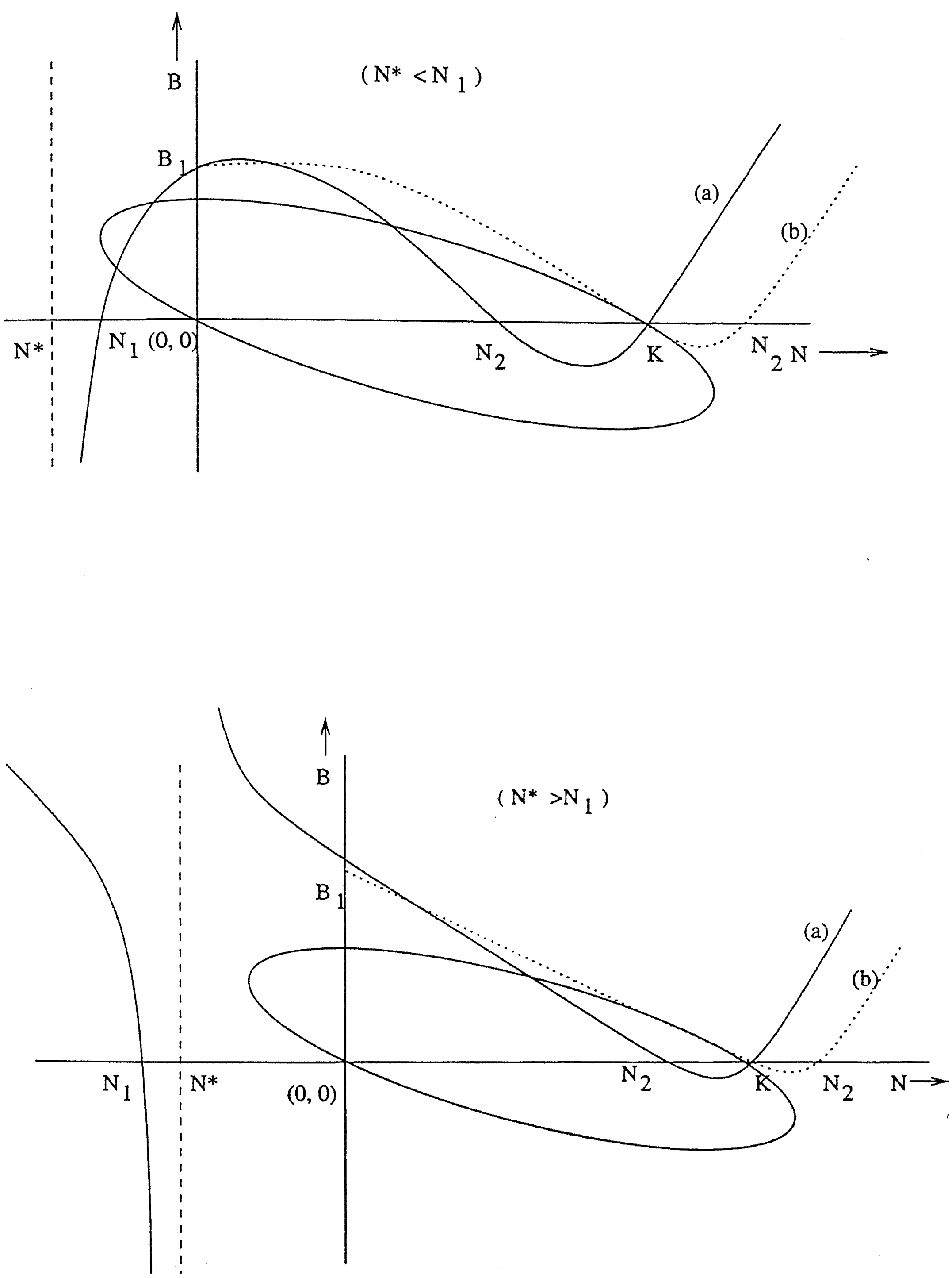


Figure 3.24: Existence of equilibrium point for $\alpha > r$.

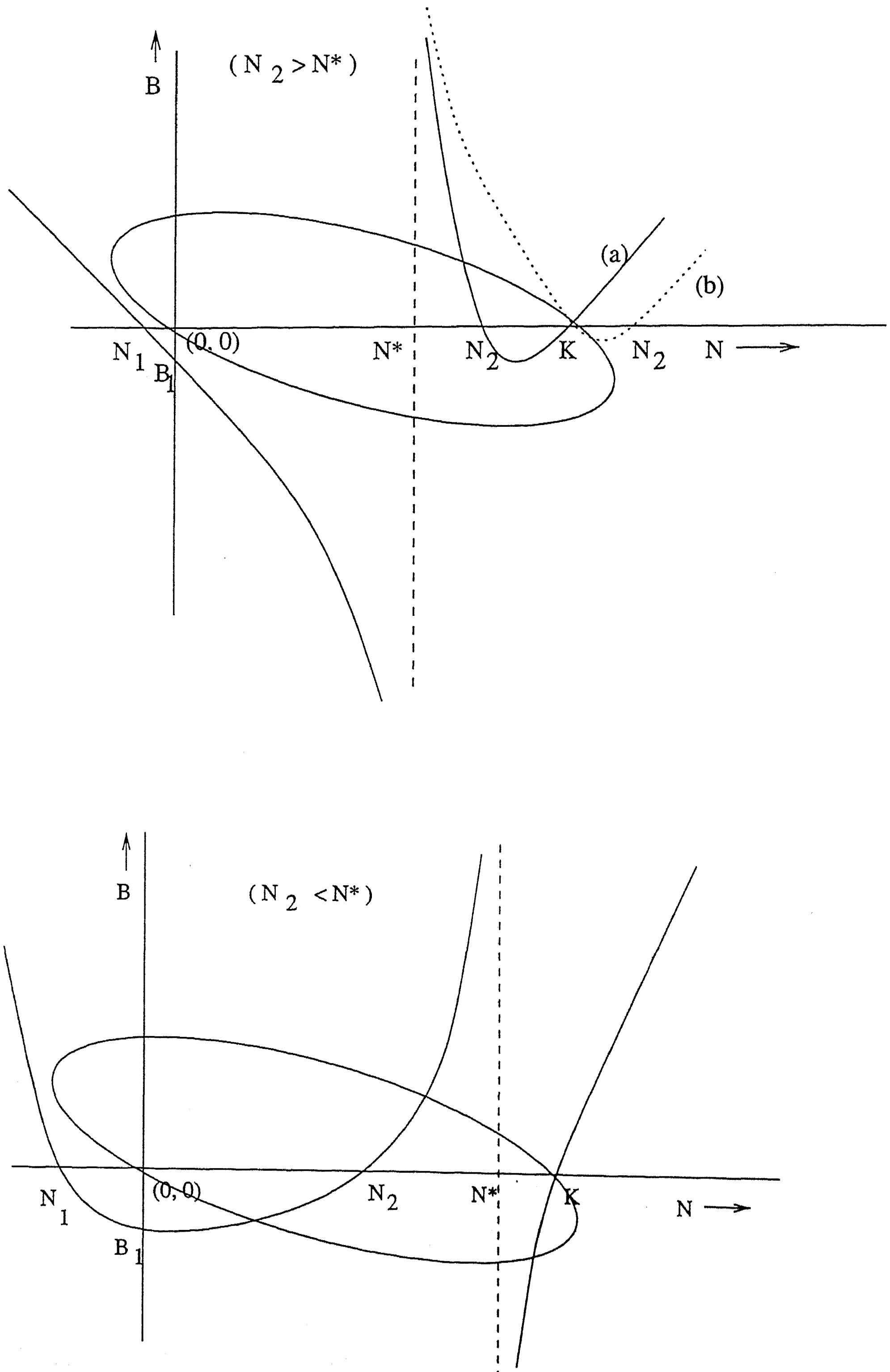


Figure 3.25: Existence of equilibrium point for $\alpha < r$.

Proof: The existence of each of the first three equilibria is obvious. We present in the following, the proof of the existence of the fourth equilibrium E_4. Setting the right hand sides of (3.27) to zero, we get the following for $N \neq 0$ and $N \neq K$,

$$Y = \frac{r}{\alpha}N(1 - \frac{N}{K}), \tag{3.28}$$

$$\frac{s}{L}B^2 - \left(s - s_0 + \delta\frac{Q_0 + lN}{\delta_0}\right)B - \frac{s_1 r}{\alpha}\left(1 - \frac{N}{K}\right)N = 0, \tag{3.29}$$

$$B = \frac{\frac{\beta r^2}{\alpha^2 K^2}(N - K)\left\{N^2 + \frac{\alpha K}{\beta r}\left(\frac{k_1}{K} - \frac{\beta r}{\alpha}\right)N - \frac{\alpha K}{\beta r}(\nu + \alpha + d)\right\}}{\frac{\lambda r}{\alpha K}\left\{N - (1 - \frac{\alpha}{r})K\right\}}, \tag{3.30}$$

where $k_1 = \beta K - (1 - a)r$.

Clearly (3.29) is an ellipse if

$$\frac{4ss_1 r}{\alpha K L} > \left(\frac{\delta l}{\delta_0}\right)^2, \tag{3.31}$$

which passes through $(0,0)$, $(K,0)$, $(0,B^*)$ and $\left(K, \frac{L}{s}\{s - s_0 + \frac{\delta}{\delta_0}(Q_0 + lK)\}\right)$. Also (3.30) is the same as (3.22). Hence as before, plotting (3.29) and (3.30) in the N-B plane, we will get a positive intersecting point $(\hat{N}, \hat{B})$ (Figs. 3.24 and 3.25) and corresponding to it we get $\hat{Y}$ and thus we have the fourth equilibrium point E_4.

3.3.2.1 Stability Analysis

Now we present the stability analysis of these equilibria. The local stability results of these equilibria are stated in the following theorem.

THEOREM 3.8 The equilibrium E_1 is unstable, the equilibrium E_2 is stable if

$$\lambda B^* + \nu + \alpha + d > r \quad and \quad \frac{(\alpha - r)}{r}\lambda B^* > \nu + \alpha + d,$$

otherwise if

$$\lambda B^* + \nu + \alpha + d < r \quad or \quad \frac{(\alpha - r)}{r}\lambda B^* < \nu + \alpha + d,$$

it is unstable. The equilibrium E_3 is unstable and E_4 is locally asymptotically stable provided $a_0 > 0$, $a_3 a_2 - a_1 > 0$ and $a_1(a_3 a_2 - a_1) - a_0\, a_3^2 > 0$, where a_0, a_1, a_2, and a_3 are given explicitly in the proof of this theorem.

Proof: The variational matrix M at (Y, N, B, E) is

$$M = \begin{pmatrix} m_{11} & m_{12} & \lambda(N-Y) & 0 \\ -\alpha & r - \frac{2r}{K}N & 0 & 0 \\ s_1 & 0 & s - s_0 + \delta E - \frac{2s}{L}B & \delta B \\ 0 & l & 0 & -\delta_0 \end{pmatrix},$$

where $m_{11} = \beta N - 2\beta Y - \lambda B - \left\{\nu + \alpha + d + (1-a)\frac{r}{K}N\right\}$, $m_{12} = \beta Y + \lambda B - (1-a)\frac{r}{K}Y$. The variational matrix M_1 at equilibrium point E_1 is given by

$$M_1 = \begin{pmatrix} -(\nu+\alpha+d) & 0 & 0 & 0 \\ -\alpha & r & 0 & 0 \\ s_1 & 0 & s - s_0 + \delta\frac{Q_0}{\delta_0} & 0 \\ 0 & l & 0 & -\delta_0 \end{pmatrix}.$$

Here two characteristic roots of the above matrix are positive implying that E_1 is unstable. The variational matrix M_2 at equilibrium point E_2 is given by

$$M_2 = \begin{pmatrix} -(\lambda B^* + \nu + \alpha + d) & \lambda B^* & 0 & 0 \\ -\alpha & r & 0 & 0 \\ s_1 & 0 & -(s - s_0 + \delta\frac{Q_0}{\delta_0}) & \delta B^* \\ 0 & l & 0 & -\delta_0 \end{pmatrix}.$$

Here two characteristic roots are $-\delta_0$ and $-(s - s_1 + \delta\frac{Q_0}{\delta_0})$, and the other roots are given by the following quadratic

$$\psi^2 + \{\lambda B^* + \nu + \alpha + d - r\}\psi - (\lambda B^* + \nu + \alpha + d)r + \lambda B^*\alpha = 0.$$

Hence using the Routh-Hurwitz criteria, the equilibrium E_2 is stable if

$$\lambda B^* + \nu + \alpha + d > r \text{ and } \lambda B^* \frac{(\alpha - r)}{r} > (\nu + \alpha + d)$$

and unstable if

$$\lambda B^* + \nu + \alpha + d < r \text{ or } \lambda B^* \frac{(\alpha - r)}{r} < (\nu + \alpha + d).$$

The variational matrix M_3 at equilibrium point E_3 is given by

$$M_3 = \begin{pmatrix} \beta K - \{\nu + \alpha + d + (1-a)r\} & 0 & \lambda K & 0 \\ -\alpha & -r & 0 & 0 \\ s_1 & 0 & (s - s_0 + \delta\frac{Q_0 + lK}{\delta_0}) & 0 \\ 0 & l & 0 & -\delta_0 \end{pmatrix}.$$

Two characteristic roots of the above matrix are $-\delta_0$ and $-r$, the other roots are given by the following quadratic equation,

$$\psi^2 - \left\{\beta K - (\nu + \alpha + d + \overline{1-a}\ r) + s - s_0 + \delta(\frac{Q_0 + lK}{\delta_0})\right\}\psi$$

$$+\left\{s - s_0 + \delta\left(\frac{Q_0 + lK}{\delta_0}\right)\right\}\left\{\beta K - (\nu + \alpha + d + \overline{1-a}\ r)\right\} - \lambda K s_1 \ = \ 0.$$

By the Routh-Hurwitz criteria the equilibrium E_3 is locally stable if

$$\beta K - (1-a)r - (\nu + \alpha + d) + s - s_0 + \delta\left(\frac{Q_0 + lK}{\delta_0}\right) < 0,$$

$$\text{and} \quad \{s - s_0 + \delta\left(\frac{Q_0 + lK}{\delta_0}\right)\}\{\beta K - (\nu + \alpha + d + \overline{1-a}\ r)\} \ > \ \lambda K s_1,$$

are satisfied and it is unstable if

$$\beta K - (1-a)r - (\nu + \alpha + d) + s - s_0 + \delta\left(\frac{Q_0 + lK}{\delta_0}\right) > 0,$$

$$\text{or} \quad \{s - s_0 + \delta\left(\frac{Q_0 + lK}{\delta_0}\right)\}\{\beta K - (\nu + \alpha + d + \overline{1-a}\ r)\} \ < \ \lambda K s_1,$$

and E_4 exists. Clearly both conditions are not satisfied simultaneously, hence E_3 is unstable.

The variational matrix M_4 at equilibrium point E_4 is given by

$$M_4 = \begin{pmatrix} -(\beta\hat{Y} + \frac{\lambda\hat{B}\hat{N}}{\hat{Y}}) & \beta\hat{Y} + \lambda\hat{B} - (1-a)\frac{r}{K}\hat{Y} & \lambda(\hat{N} - \hat{Y}) & 0 \\ -\alpha & r - \frac{2r}{K}\hat{N} & 0 & 0 \\ s_1 & 0 & -\left(\frac{s\hat{B}}{L} + \frac{s_1\hat{Y}}{\hat{B}}\right) & \delta\hat{B} \\ 0 & l & 0 & -\delta_0 \end{pmatrix}.$$

The characteristic polynomial in this case is given by

$$\psi^4 + a_3\ \psi^3 + a_2\ \psi^2 + a_1\psi + a_0 = 0,$$

where

$$a_3 \ = \ \left[\beta\hat{Y} + \frac{\lambda\hat{B}\hat{N}}{\hat{Y}}\right] + \left[-r + \frac{2r}{K}\hat{N}\right] + \left[\frac{s}{L}\hat{B} + \frac{s_1\hat{Y}}{\hat{B}}\right] + \delta_0 > 0,$$

$$a_2 \ = \ \delta_0\left\{\beta\hat{Y} + \frac{\lambda\hat{B}\hat{N}}{\hat{Y}} + \frac{2r\hat{N}}{K} - r + \frac{s\hat{B}}{L} + \frac{s_1\hat{Y}}{\hat{B}}\right\} + \left(\beta\hat{Y} + \frac{\lambda\hat{B}\hat{N}}{\hat{Y}}\right)\left(\frac{2r\hat{N}}{K} - r + \frac{s\hat{B}}{L} + \frac{s_1\hat{Y}}{\hat{B}}\right)$$

$$
\begin{aligned}
&+\left(\frac{2r\hat{N}}{K}-r\right)\left(\frac{s\hat{B}}{L}+\frac{s_1\hat{Y}}{\hat{B}}\right)+\alpha\left\{\beta\hat{Y}+\lambda\hat{B}-(1-a)\frac{r}{K}\hat{Y}\right\}-s_1\lambda(\hat{N}-\hat{Y})\\
=\ &\left[\delta_0\left(\beta\hat{Y}+\frac{\lambda\hat{B}\hat{N}}{\hat{Y}}\right)\right]+\left[\delta_0\left(\frac{2r\hat{N}}{K}-r\right)\right]+\left[\delta_0\left(\frac{s\hat{B}}{L}+\frac{s_1\hat{Y}}{\hat{B}}\right)\right]\\
&+\left[\left(\beta\hat{Y}+\frac{\lambda\hat{B}\hat{N}}{\hat{Y}}\right)\left(\frac{2r\hat{N}}{K}-r\right)\right]+\left[\left(\frac{2r\hat{N}}{K}-r\right)\left(\frac{s\hat{B}}{L}+\frac{s_1\hat{Y}}{\hat{B}}\right)\right]\\
&+\left[\alpha\left\{\beta\hat{Y}+\lambda\hat{B}-(1-a)\frac{r}{K}\hat{Y}\right\}\right]+\left(\beta\hat{Y}+\frac{\lambda\hat{B}\hat{N}}{\hat{Y}}\right)\frac{s\hat{B}}{L}+\beta\hat{Y}\frac{s_1\hat{Y}}{\hat{B}}+s_1\lambda\hat{Y},
\end{aligned}
$$

$$
\begin{aligned}
a_1 =\ &\delta_0\left(\beta\hat{Y}+\frac{\lambda\hat{B}\hat{N}}{\hat{Y}}\right)\left(\frac{2r\hat{N}}{K}-r\right)+\left(\beta\hat{Y}+\frac{\lambda\hat{B}\hat{N}}{\hat{Y}}\right)\left(\frac{2r\hat{N}}{K}-r\right)\left(\frac{s\hat{B}}{L}+\frac{s_1\hat{Y}}{\hat{B}}\right)\\
&+\delta_0\left(\frac{2r\hat{N}}{K}-r\right)\left(\frac{s\hat{B}}{L}+\frac{s_1\hat{Y}}{\hat{B}}\right)+\delta_0\left(\frac{s\hat{B}}{L}+\frac{s_1\hat{Y}}{\hat{B}}\right)\left(\beta\hat{Y}+\frac{\lambda\hat{B}\hat{N}}{\hat{Y}}\right)\\
&+\alpha\left\{\beta\hat{Y}+\lambda\hat{B}-(1-a)\frac{r}{K}\hat{Y}\right\}\left(\delta_0+\frac{s\hat{B}}{L}+\frac{s_1\hat{Y}}{\hat{B}}\right)-s_1\lambda(\hat{N}-\hat{Y})\left\{\delta_0+\frac{2r\hat{N}}{K}-r\right\}
\end{aligned}
$$

$$
\begin{aligned}
a_0 =\ &\delta_0\left(\beta\hat{Y}+\frac{\lambda\hat{B}\hat{N}}{\hat{Y}}\right)\left(\frac{2r\hat{N}}{K}-r\right)\left(\frac{s\hat{B}}{L}+\frac{s_1\hat{Y}}{\hat{B}}\right)\\
&+\alpha\delta_0\left\{\beta\hat{Y}+\lambda\hat{B}-(1-a)\frac{r}{K}\hat{Y}\right\}\left(\frac{s\hat{B}}{L}+\frac{s_1\hat{Y}}{\hat{B}}\right)\\
&-s_1\delta_0\lambda(\hat{N}-\hat{Y})\left(\frac{2r\hat{N}}{K}-r\right)+\alpha\lambda l\delta\hat{B}(\hat{N}-\hat{Y}).
\end{aligned}
$$

By Murata (1977), conditions for local stability of the system are

$$
a_3>0,\quad \begin{vmatrix} a_3 & a_1\\ 1 & a_2\end{vmatrix}>0,\quad \begin{vmatrix} a_3 & a_1 & 0\\ 1 & a_2 & a_0\\ 0 & a_3 & a_1\end{vmatrix}>0,\quad \begin{vmatrix} a_3 & a_1 & 0 & 0\\ 1 & a_2 & a_0 & 0\\ 0 & a_3 & a_1 & 0\\ 0 & 1 & a_2 & a_0\end{vmatrix}>0.
$$

First inequality is obviously true, so if other inequalities are satisfied, then this equilibrium is locally asymptotically stable.

Remark: We note that the second condition is satisfied for $\hat{N} > \frac{K}{2}$. So for simulation we choose set of parameters such that $\hat{N} > \frac{K}{2}$.

Nonlinear Analysis and Simulation:

As before, we speculate that the system (3.27) may be globally stable for $\hat{N} > \frac{K}{2}$. To show this, the system (3.27) is integrated by the fourth order Runge-Kutta method using

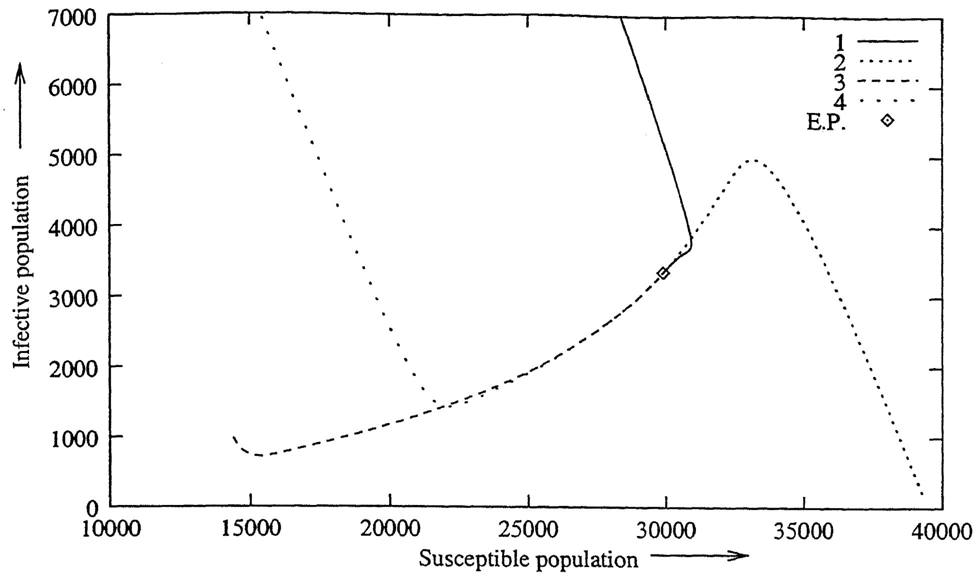


Figure 3.26: Variation of infective population with susceptible population.

same set of parameters as given in Case I, with $Q_0 = Q_a$ and an additional parameter $l = 0.000005$, which satisfies the local stability conditions.

The equilibrium values for this set of parameters are determined as:

$$\hat{Y} = 3359.500, \quad \hat{N} = 33267.756, \quad \hat{B} = 2034237.432, \quad \hat{E} = 20166.338.$$

Simulation is performed for different initial positions 1, 2, 3 and 4 shown in Fig. 3.26. From this figure, it is clear that this equilibrium is globally stable provided that we start away from the other equilibria. Also effects of various parameters such as s, s_1, δ, L, l and r, on the infective population are shown in Figs. 3.27-3.32. As before it is concluded that due to increase in any of these parameters, the infective population increases as expected.

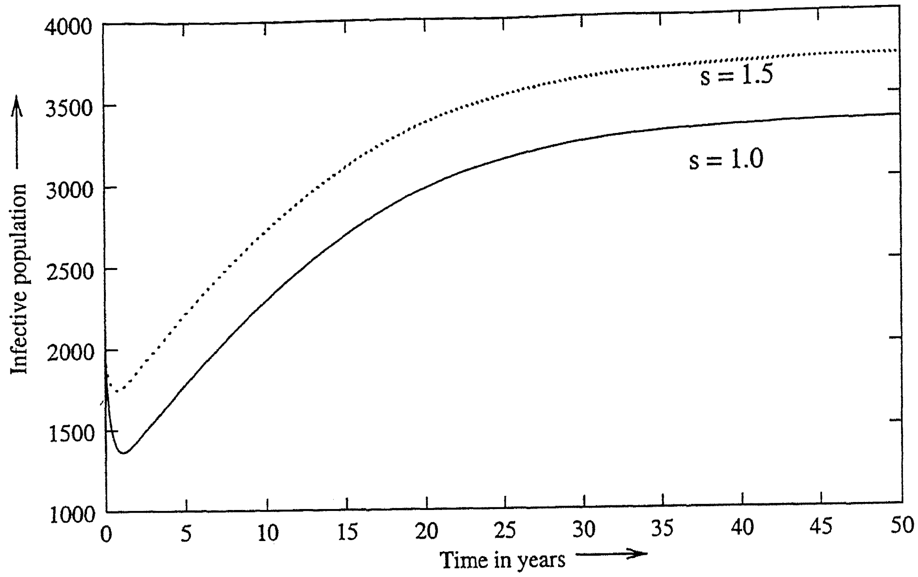


Figure 3.27: Variation of infective population with time for different intrinsic growth rate of bacteria population.

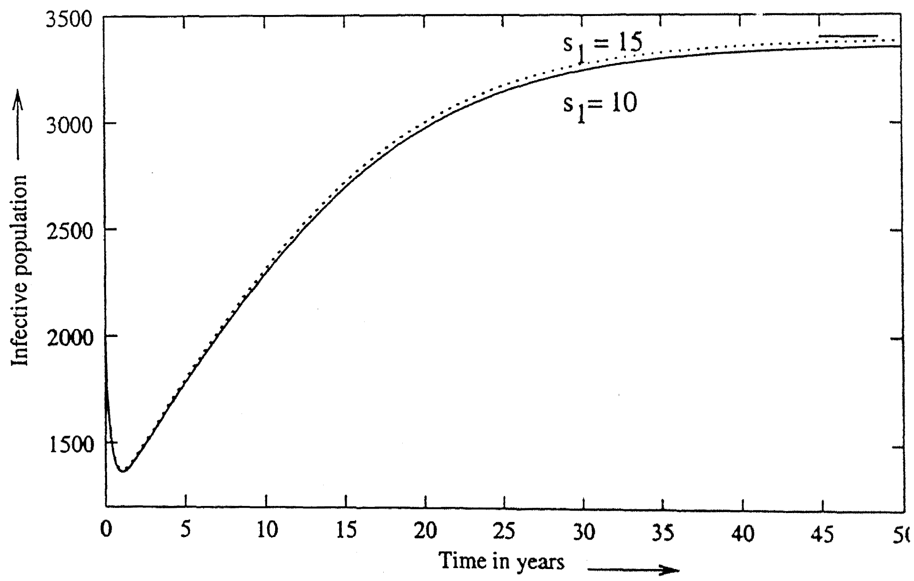


Figure 3.28: Variation of infective population with time for different growth rate of bacteria population due to infective human population.

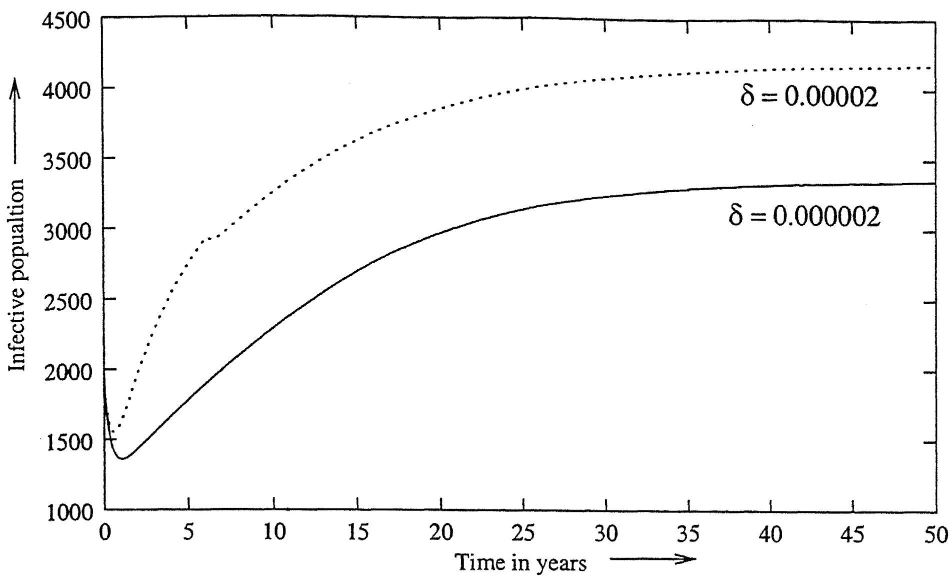


Figure 3.29: Variation of infective population with time for different growth rate of bacteria population corresponding to environmental discharges.

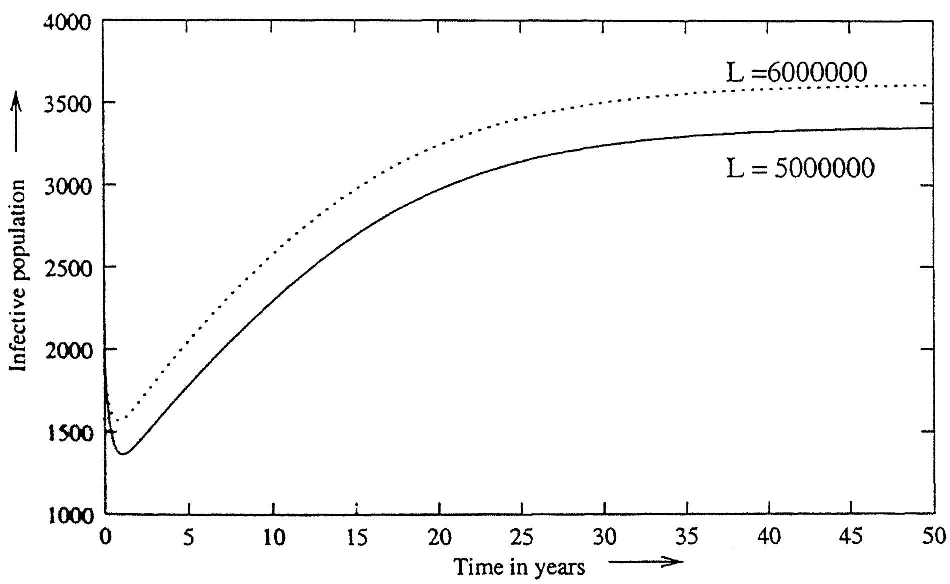


Figure 3.30: Variation of infective population with time for different carrying capacity of bacteria population.

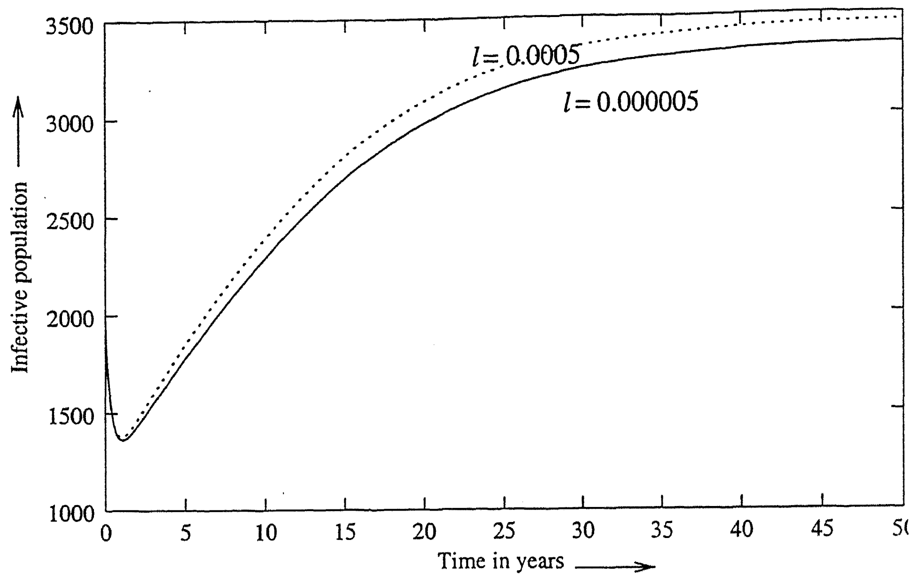


Figure 3.31: Variation of infective population with time for different l.

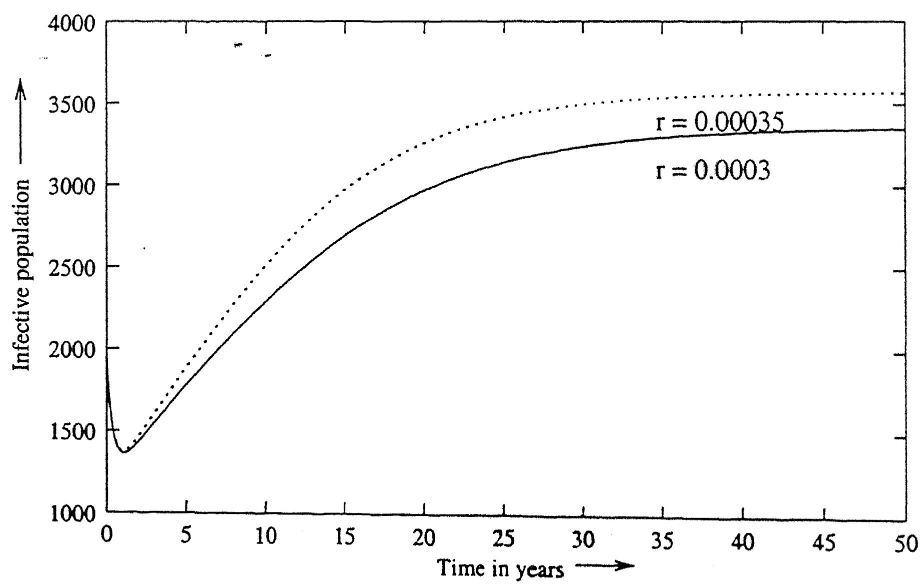


Figure 3.32: Variation of infective population with time for different growth rate of human population.

3.4 Conclusions

In this chapter, the SIS model for bacterial infectious diseases, like tuberculosis, typhoid, etc., caused by direct contact of susceptibles with infectives as well as by bacteria is proposed and analyzed. When demography for human population is constant immigration, it is shown that the endemic equilibria is globally stable when the cumulative rate of environmental discharges conducive to the growth of bacterial population is a constant. When the cumulative rate of the environmental discharges is a function of total human population, the endemic equilibria is also locally and globally stable under same conditions. The later result is shown by computer simulation. For logistic growth of human population, in each case the nontrivial equilibrium is locally asymptotically stable under certain conditions. By simulation it is shown that nontrivial equilibrium is globally stable under local stability conditions for that set of parameters. It is concluded from the analysis that the growth of bacteria caused by conducive environmental discharge due to human sources increases the spread of the infectious disease. It is noted that if the population increases either by immigration or logistic growth, the spread of the bacterial disease further increases and it becomes more endemic.

Chapter 4

Modelling the Spread of Malaria: Environmental and Demographic Effects

4.1 Introduction

It is well known that the spread of malaria is governed by the following factors: (i) the density of the human population and its rate of growth, (ii) the density of the mosquito population and its rate of growth, (iii) various environmental factors such as rain, temperature, humidity and so on, (iv) ecological factors such as vegetation, biomass, cattle population, etc. and (v) geographical factors. In most of the tropical countries including India, the emergence of malaria has taken place and it has become endemic in the North-Eastern part of India, where this disease is spread by a lethal parasite called Plasmodium Falciparum. Although there have been several experimental studies related to surveys of malaria in different regions (Sharma 1991, 1998, Das 1991), the study of the spread of disease using mathematical models by considering the factors mentioned above has not been conducted, particularly when the densities of the human and mosquito populations are variable. However, in the case of these populations being constants, Bailey (1979) has given a simple mathematical model by considering criss-cross interaction between female mosquitoes and the human population. Some other studies for the spread of malaria have

also been conducted by proposing both stochastic and deterministic models (Radcliffe 1973, 1974, Dietz et al. 1974, Molineaux et al. 1978, Bailey 1979, 1982). It has been suggested that the malaria model and the Gonorrhea model are very similar in mathematical structure (Nallaswamy and Shukla 1982). In those models the environmental and ecological factors have not been taken into account, though these factors play significant roles in the spread of malaria. In this chapter, therefore, an SIS non-linear mathematical model is proposed and analyzed to study the effects of household and other environmental discharges on the ground, such as waste water, food stuff and so on, which are caused by the human population. It is assumed that the density of the mosquito population follows a generalized logistic model such that its growth rate decreases but its death rate increases as population density increases towards its carrying capacity with respect to the environment. It is further assumed that the growth rate of the mosquito population increases as the density of cumulative environmental discharges increases. The human population density is divided into susceptible and infective classes. The mosquito population density is also divided into susceptible mosquitoes and infective mosquitoes. Our main focus is to investigate the effect of cumulative household discharges conducive to the growth of the mosquito population, on the spread of malaria by considering the following two types of demographics for the human population:

(i) a population with constant immigration,

(ii) a population with a logistic growth rate.

4.2 Malaria Model with Immigration

We consider here an SIS model, where the human population density $N_1(t)$ is divided into two classes namely, the susceptible class $X_1(t)$ and the infective class $Y_1(t)$. The mosquito population density $N_2(t)$ is divided into the susceptible class $X_2(t)$ and the infective class $Y_2(t)$. Keeping in view the above and by considering the criss-cross interaction of the mosquito population with the human population, a model can be written as follows:

$$
\begin{aligned}
\dot{X}_1 &= A - d_1 X_1 - \beta_1 X_1 Y_2 + \nu_1 Y_1, \\
\dot{Y}_1 &= \beta_1 X_1 Y_2 - (\nu_1 + \alpha_1 + d_1) Y_1, \\
\dot{N}_1 &= A - d_1 N_1 - \alpha_1 Y_1, \\
\dot{X}_2 &= (b_2 - a' \frac{r_2}{K_2} N_2) N_2 - \{d_2 + (1 - a') \frac{r_2}{K_2} N_2\} X_2 - \beta_2 X_2 Y_1 - \alpha_2 X_2 + \delta_2 N_2 E, \\
\dot{Y}_2 &= \beta_2 X_2 Y_1 - \{\alpha_2 + d_2 + (1 - a') \frac{r_2}{K_2} N_2\} Y_2, \\
\dot{N}_2 &= r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \alpha_2 N_2 + \delta_2 N_2 E, \\
\dot{E} &= Q(N_1) - \delta_0 E, \\
N_1 &= X_1 + Y_1, \quad N_2 = X_2 + Y_2, \\
X_1(0) &= X_{10} > 0, \; Y_1(0) = Y_{10} \geq 0, \; X_2(0) = X_{20} \geq 0, \; Y_{20} \geq 0, \; E(0) = E_0 > 0.
\end{aligned}
\tag{4.1}
$$

In model (4.1), A is the constant immigration rate of the human population; d_1 is the natural death rate constant; β_1 is the interaction coefficient of the susceptible human with the infective mosquito population; ν_1 is the recovery rate coefficient of the human population; α_1 is the disease related death rate constant; b_2 and d_2 are the birth and the death rate constants corresponding to the mosquito population; $r_2 = b_2 - d_2$ is the growth rate coefficient of the mosquito population; K_2 is the carrying capacity of the mosquito population in the natural environment; α_2 is the death rate of mosquitoes due to control measures ($r_2 > \alpha_2$); β_2 is the interaction coefficient of susceptible mosquitoes with the infective human class; δ_2 is the growth rate coefficient of the mosquito population due to the environmental discharges of cumulative concentration E; Q is the cumulative rate of environmental discharges which is human population density dependent; δ_0 is its cumulative depletion rate and $0 \leq a' \leq 1$ is a constant (Gao et al. 1992), which governs the logistic birth and logistic death of the mosquito population.

We analyze the model (4.1) for the following two cases :

(i) the rate of cumulative environmental discharges Q is a constant, and

(ii) the rate of cumulative environmental discharges Q is a function of human population density. We consider the form of $Q(N_1)$ as $Q(N_1) = Q_0 + lN_1$, where l is a constant.

4.2.1 Case I: $Q = Q_a$ a Constant

Since $X_1 + Y_1 = N_1$ and $X_2 + Y_2 = N_2$, in this case it is sufficient to consider the following subsystem of the model (4.1),

$$
\begin{aligned}
\dot{Y_1} &= \beta_1(N_1 - Y_1)\, Y_2 - (\nu_1 + \alpha_1 + d_1)Y_1, \\
\dot{N_1} &= A - d_1 N_1 - \alpha_1\, Y_1, \\
\dot{Y_2} &= \beta_2\,(N_2 - Y_2)\, Y_1 - \left\{\alpha_2 + d_2 + (1 - a')\frac{r_2}{K_2} N_2\right\}\, Y_2, \\
\dot{N_2} &= r_2\, N_2 \left(1 - \frac{N_2}{K_2}\right) - \alpha_2\, N_2 + \delta_2\, N_2\, E, \\
\dot{E} &= Q_a - \delta_0\, E.
\end{aligned}
\tag{4.2}
$$

We see that the region of attraction of the system (4.2)

$$
\begin{aligned}
T &= \left\{(Y_1, N_1, Y_2, N_2, E) : 0 \leq Y_1 \leq N_1 \leq \frac{A}{d_1},\ 0 \leq Y_2 \leq N_2 \leq \frac{K_2}{r_2}\left(r_2 - \alpha_2 + \delta_2 \frac{Q_a}{\delta_0}\right),\right. \\
&\qquad \left. 0 \leq\ E \leq \frac{Q_a}{\delta_0}\right\},
\end{aligned}
$$

is positively invariant and all solutions starting in this region T stay in T. The continuity of right sides of (4.2) and their derivatives imply that a unique solution exists (Hale 1969). Since the system (4.2) is autonomous, the effects of N_2 and E on the spread of malaria can be qualitatively studied by taking their asymptotic values as $t \to \infty$ in the last two equations of (4.2). Thus we have

$$\lim_{t\to\infty} \sup E(t) = \frac{Q_a}{\delta_0} = \bar{E} \quad \text{(say)}$$

$$\text{and if } N_2(0) > 0 \quad \lim_{t\to\infty} \sup N_2 = \frac{K_2}{r_2}(r_2 - \alpha_2 + \delta_2 \bar{E}) = \bar{N_2} \quad \text{(say)}.$$

Here it is noted that $\bar{N_2}$ increases as Q_a and K_2 increase or as α_2 decreases showing the effect of the growth of mosquito population. Now it suffices to study the global behavior of the system (4.2) by the following system of equations:

$$
\begin{aligned}
\dot{Y_1} &= \beta_1(N_1 - Y_1)\, Y_2 - (\nu_1 + \alpha_1 + d_1)Y_1, \\
\dot{N_1} &= A - d_1 N_1 - \alpha_1\, Y_1, \\
\dot{Y_2} &= \beta_2\,(\bar{N_2} - Y_2)\, Y_1 - \{\alpha_2 + d_2 + (1 - a')\frac{r_2}{K_2}\bar{N_2}\}\, Y_2.
\end{aligned}
\tag{4.3}
$$

The result of an equilibrium analysis is stated in the following theorem. The proof is obvious.

THEOREM 4.1 There exist following two equilibria, namely
(i) $E_0\left(0,\ \frac{A}{d_1},\ 0\right)$ *and (ii)* $E_1(\hat{Y}_1,\ \hat{N}_1,\ \hat{Y}_2)$, *where*

$$\hat{Y}_1 = \frac{\beta_1 \frac{A}{d_1} \hat{Y}_2}{\beta_1(1+\frac{\alpha_1}{d_1})\hat{Y}_2 + \nu_1 + \alpha_1 + d_1}, \quad \hat{N}_1 = \frac{A - \alpha_1 \hat{Y}_1}{d_1},$$

and

$$\hat{Y}_2 = \frac{\beta_1\beta_2 \frac{A}{d_1}\bar{N}_2 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}(\nu_1+\alpha_1+d_1)}{\beta_1\left[\beta_2\frac{A}{d_1} + \left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\left(1+\frac{\alpha_1}{d_1}\right)\right]},$$

E_1 *exists if*

$$\frac{\beta_1\,\beta_2\,\dfrac{A}{d_1}\,\bar{N}_2}{\left\{\alpha_2 + d_2 + (1-a')\dfrac{r_2}{K_2}\bar{N}_2\right\}(\nu_1+\alpha_1+d_1)} = R_0 \ \ (say) \ > \ 1. \tag{4.4}$$

It is easy to see that R_0 *is the threshold parameter of the system (4.3).*

Remark: We note here that $\hat{Y}_2$ increases with $\bar{N}_2$. Hence $\hat{Y}_1$ increases as Q_a or K_2 increases, i.e. the infective human population increases as the cumulative rate of environmental discharges increases.

4.2.1.1 Stability Analysis

In the following, we study the linear analysis of these equilibria and nonlinear analysis of the nontrivial equilibrium E_1. We state the local stability of these two equilibria in the following theorem.

THEOREM 4.2 (i) The equilibrium point E_0 *is locally asymptotically stable if* $R_0 < 1$, *otherwise if* $R_0 > 1$ *it is unstable and then the second equilibrium* E_1 *exists.*
(ii) The second equilibrium E_1*, if it exists, is locally asymptotically stable.*

Proof: To study the local stability of these equilibria, we find the variational matrix corresponding to system (4.3) as

$$M = \begin{pmatrix} -(\beta_1\, Y_2 + \nu_1 + \alpha_1 + d_1) & \beta_1\, Y_2 & \beta_1\,(N_1 - Y_1) \\ -\alpha_1 & -d_1 & 0 \\ \beta_2(\bar{N}_2 - Y_2) & 0 & -[\beta_2\, Y_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}] \end{pmatrix}.$$

At the equilibrium $E_0\left(0,\ \frac{A}{d_1},\ 0\right)$, the variational matrix M_0 is given by

$$M_0 = \begin{pmatrix} -(\nu_1 + \alpha_1 + d_1) & 0 & \beta_1\,\frac{A}{d_1} \\ -\alpha_1 & -d_1 & 0 \\ \beta_2\bar{N}_2 & 0 & -\left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\} \end{pmatrix}.$$

The characteristic polynomial corresponding to matrix M_0 is

$$(\psi + d_1)\left[\psi^2 + \left(\nu_1 + \alpha_1 + d_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right)\psi \right.$$
$$\left. +(\nu_1 + \alpha_1 + d_1)\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\} - \beta_1\beta_2\frac{A}{d_1}\bar{N}_2\right] = 0.$$

Using the Routh-Hurwitz criteria in the above quadratic we note that the equilibrium $E_0(0,\ \frac{A}{d_1},\ 0)$ is locally asymptotically stable if $R_0 < 1$. Further, it is unstable if $R_0 > 1$ and in this case the second endemic equilibrium point exists.

At the equilibrium $E_1(\hat{Y}_1,\ \hat{N}_1,\ \hat{Y}_2)$, the variational matrix $\hat{M}$ is given by

$$\hat{M} = \begin{pmatrix} -(\beta_1\, \hat{Y}_2 + \nu_1 + \alpha_1 + d_1) & \beta_1\, \hat{Y}_2 & \beta_1\,(\hat{N}_1 - Y_1) \\ -\alpha_1 & -d_1 & 0 \\ \beta_2(\bar{N}_2 - \hat{Y}_2) & 0 & -[\beta_2\, \hat{Y}_1 + \{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\}] \end{pmatrix}.$$

The characteristic polynomial corresponding to matrix $\hat{M}$ is

$$\psi^3 + b_1\,\psi^2 + b_2\,\psi + b_3 = 0,$$

where

$$\begin{aligned} b_1 &= \beta_1\,\hat{Y}_2 + \nu_1 + \alpha_1 + 2d_1 + \beta_2\,\hat{Y}_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2, \\ b_2 &= (\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1)(d_1 + \beta_2\,\hat{Y}_1) + \beta_1\hat{Y}_2\left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\} \\ &\quad +\alpha_1\beta_1\hat{Y}_2 + d_1\left[\beta_2\hat{Y}_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\right], \\ b_3 &= \beta_1\hat{Y}_2(\alpha_1 + d_1)\left[\beta_2\hat{Y}_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\right] + \beta_2\hat{Y}_1 d_1(\nu_1 + \alpha_1 + d_1) \end{aligned}$$

We note here that $b_1 > 0$ and also $b_1 b_2 - b_3 > 0$. Hence by the Routh-Hurwitz criteria the equilibrium $E_1(\hat{Y}_1, \hat{N}_1, \hat{Y}_2)$, if it exists, is locally asymptotically stable.

Nonlinear Analysis and Simulation: Before proceeding to simulation, we first analyze the model (4.3) with the disease related death rate $\alpha_1 = 0$. In this case the model (4.3) reduces to

$$\begin{aligned} \dot{Y}_1 &= \beta_1(N_1 - Y_1)\, Y_2 - (\nu_1 + d_1)Y_1, \\ \dot{N}_1 &= A - d_1 N_1, \\ \dot{Y}_2 &= \beta_2(\bar{N}_2 - Y_2)\, Y_1 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\} Y_2. \end{aligned} \tag{4.5}$$

The equilibria of the system (4.5) are $E_0^*\left(0, \frac{A}{d_1}, 0\right)$ and $E_1^*(\hat{Y}_1, \hat{N}_1, \hat{Y}_2)$, where

$$\hat{Y}_1 = \frac{\beta_1 \hat{N}_1 \hat{Y}_2}{\beta_1 \hat{Y}_2 + \nu_1 + d_1}, \quad \hat{N}_1 = \frac{A}{d_1}, \quad \hat{Y}_2 = \frac{\beta_1 \beta_2 \hat{N}_1 \bar{N}_2 - (\nu_1 + d_1)\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\}}{\beta_1[\beta_2 \hat{N}_1 + \{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\}]}. \tag{4.6}$$

The second equilibrium exists if

$$\frac{\beta_1 \beta_2 \hat{N}_1 \bar{N}_2}{(\nu_1 + d_1)\,\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\}} = R_{00} \ \text{(say)} \ > 1. \tag{4.7}$$

We note from (4.6) that $\hat{Y}_2$ increases as $\bar{N}_2$, i.e. as Q_a increases. This leads to an increase in $\hat{Y}_1$ and for large $\hat{Y}_2$, $\hat{Y}_1 \to \hat{N}_1$. It is easy to see that the equilibrium $E_0^*(0, \frac{A}{d_1}, 0)$ is unstable under (4.7) and is stable otherwise. Here R_{00} is the threshold for the system of equations (4.5).

We can also prove that the equilibrium $E_1^*(\hat{Y}_1, \hat{N}_1, \hat{Y}_2)$, if it exists, is globally stable, by using the following Liapunov function and choosing suitable values of k_1 and k_2,

$$V = \frac{1}{2}\,(Y_1 - \hat{Y}_1)^2 + k_1\frac{1}{2}\,(N_1 - \hat{N}_1)^2 + k_2\frac{1}{2}\,(Y_2 - \hat{Y}_2)^2.$$

Since our system (4.3) is bounded by the system (4.5) in the region T, using a comparison theorem (Lakshmikantham and Leela 1969), we conclude that the solution of system (4.3) is bounded by the solution of (4.3) with $\alpha_1 = 0$. Thus we conjecture that the equilibrium E_1 may be globally stable. This result is illustrated by integrating system (4.3) using

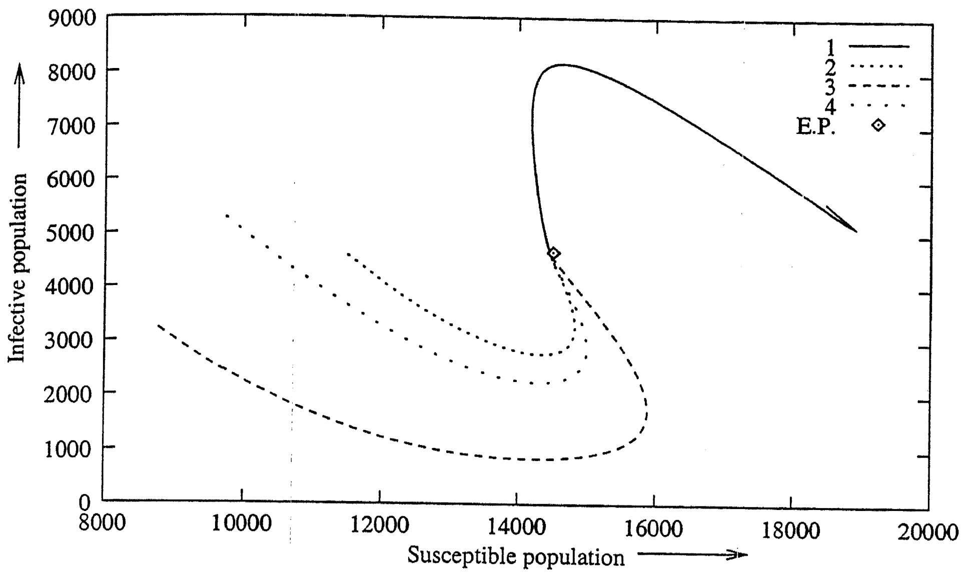


Figure 4.1: Variation of infective population Y_1 with susceptible population X_1.

the fourth order Runge-Kutta method in the region T. The following data set is used (Greenhalgh 1992),

$$\beta_1 = 0.00000025 = \beta_2,\ \nu_1 = 0.012,\ \alpha_1 = 0.0005,\ A = 10,\ d_1 = 0.0004,\ \alpha_2 = 0.045,\ r_2 = 1,$$

$$d_2 = 0.02,\ a' = 0.999,\ K_2 = 1000000,\ Q_a = 20,\ \delta_0 = 0.001,\ \delta_2 = 0.0000002.$$

Under the above parameter values, the equilibrium point E_1 is found as:

$$\hat{Y}_1 = 4690.018,\quad \hat{N}_1 = 19137.477,\quad \hat{Y}_2 = 16750.691.$$

The simulation is performed for different initial positions 1, 2, 3 and 4 as given below,

1... $Y_1(0) = 5610,\quad N_1(0) = 24092,\quad Y_2(0) = 2180.$

2... $Y_1(0) = 4610,\quad N_1(0) = 16092,\quad Y_2(0) = 1180.$

3... $Y_1(0) = 3222,\quad N_1(0) = 12000,\quad Y_2(0) = 1000.$

4... $Y_1(0) = 5300,\quad N_1(0) = 15000,\quad Y_2(0) = 90.$

In Fig. 4.1, the infective population is plotted against the susceptible population which is consistent with the solution tending to E_1 if the starting point is not E_0. Thus the equilibrium point E_1 appears to be globally stable if we start away from E_0. Also in Fig. 4.2, the infective population is plotted with time for different A and Q_a and we note that $\hat{Y}_1$ increases as A or Q_a increases showing that the spread of malaria increases and it becomes more endemic due to increased immigration and household discharges.

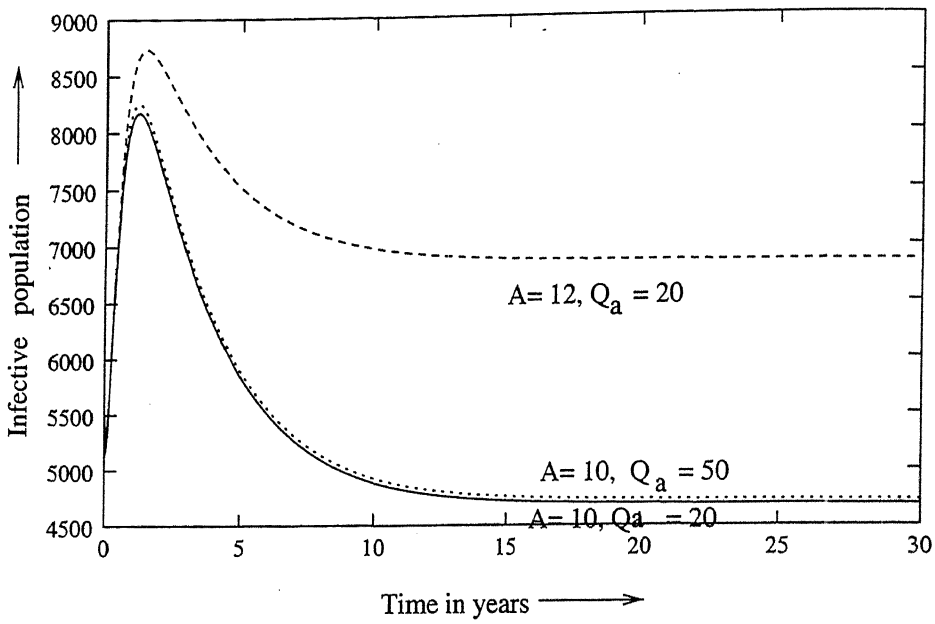


Figure 4.2: Variation of infective population Y_1 with time for different immigration rates of human population and the rate of cumulative environmental discharges.

4.2.2 Case II: $Q = Q_0 + lN_1$

In this case using $X_1 + Y_1 = N_1$, $X_2 + Y_2 = N_2$ and $Q = Q_0 + lN_1$, the model (4.1) can be written as follows,

$$\begin{aligned}
\dot{Y_1} &= \beta_1 (N_1 - Y_1) Y_2 - (\nu_1 + \alpha_1 + d_1) Y_1, \\
\dot{N_1} &= A - d_1 N_1 - \alpha_1 Y_1, \\
\dot{Y_2} &= \beta_2 (N_2 - Y_2) Y_1 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2} N_2\right\} Y_2, \\
\dot{N_2} &= r_2 N_2 \left(1 - \frac{N_2}{K_2}\right) - \alpha_2 N_2 + \delta_2 N_2 E, \\
\dot{E} &= Q_0 + l N_1 - \delta_0 E.
\end{aligned} \tag{4.7}$$

We see that in this case the region of attraction is

$$\begin{aligned}
T' &= \left\{(Y_1, N_1, Y_2, N_2, E) : 0 \le Y_1 \le N_1 \le \frac{A}{d_1},\right. \\
&\quad \left. 0 \le Y_2 \le N_2 \le \frac{K_2}{r_2}\left(r_2 - \alpha_2 + \delta_2 \frac{Q_0 + l\frac{A}{d_1}}{\delta_0}\right), 0 \le E \le \frac{Q_0 + l\frac{A}{d_1}}{\delta_0}\right\},
\end{aligned}$$

and the model is well posed. The result of equilibrium analysis is stated in the following theorem.

THEOREM 4.3 There exist the following three equilibria, namely

(i) $\bar{E}_1\left(0,\ \frac{A}{d_1},\ 0,\ 0,\ \frac{Q_0+\frac{lA}{d_1}}{\delta_0}\right)$,

(ii) $\bar{E}_2\left(0,\ \frac{A}{d_1},\ 0,\ \bar{\bar{N}}_2,\ \frac{Q_0+\frac{lA}{d_1}}{\delta_0}\right)$, *where* $\bar{\bar{N}}_2=\frac{K_2}{r_2}\left\{r_2-\alpha_2+\delta_2\left(\frac{Q_0+l\frac{A}{d_1}}{\delta_0}\right)\right\}$

and (iii) $\bar{E}_3\left(\hat{Y}_1,\ \hat{N}_1,\ \hat{Y}_2,\ \hat{N}_2,\ \frac{Q_0+l\,\hat{N}_1}{\delta_0}\right)$. *$E_3$ exists if*

$$\frac{\beta_1\beta_2\frac{A}{d_1}\bar{\bar{N}}_2}{[\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{\bar{N}}_2](\nu_1+\alpha_1+d_1)}=R_0'\ (say)\ >\ 1.$$

Proof: The existence of either of the first two equilibria is obvious. The existence of the third equilibrium $\bar{E}_3$ is shown by the isocline method. Setting the right hand sides of the system of equations (4.8) to zero, we get the following set of equations,

$$E=\frac{Q_0+lN_1}{\delta_0},\quad N_2=\frac{K_2}{r_2}\left\{r_2-\alpha_2+\delta_2(\frac{Q_0+lN_1}{\delta_0})\right\},\tag{4.9}$$

$$Y_2=\frac{\beta_2N_2Y_1}{\beta_2Y_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}N_2},\quad N_1=\frac{A-\alpha_1Y_1}{d_1},\ Y_1=\frac{\beta_1(N_1-Y_1)Y_2}{\nu_1+\alpha_1+d_1}.\tag{4.10}$$

Using these equations, we have the following quadratic in Y_1,

$$F(Y_1)=p_0Y_1^2+p_1Y_1+p_2=0,\tag{4.11}$$

where

$$\begin{aligned}
p_0 &= \beta_1\beta_2\left(1+\frac{\alpha_1}{d_1}\right)\frac{\delta_2l\alpha_1}{\delta_0d_1}\frac{K_2}{r_2},\\
p_1 &= -\left[\beta_1\beta_2\frac{A}{d_1}\frac{\delta_2l\alpha_1K_2}{\delta_0d_1r_2}+\beta_1\beta_2(1+\frac{\alpha_1}{d_1})\bar{\bar{N}}_2+(\nu_1+\alpha_1+d_1)\left\{\beta_2-(1-a')\frac{\delta_2l\alpha_1}{\delta_0d_1}\right\}\right],\\
p_0 &= \left[\beta_1\beta_2\frac{A}{d_1}\bar{\bar{N}}_2-(\nu_1+\alpha_1+d_1)\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{\bar{N}}_2\right\}\right].
\end{aligned}$$

From (4.11), we have the following,

(i) $F(0)=\beta_1\beta_2\frac{A}{d_1}\bar{\bar{N}}_2-\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{\bar{N}}_2\}(\nu_1+\alpha_1+d_1)$,

(ii) $F\left(\frac{A}{\alpha_1+d_1}\right)<0$, and

(iii) $F(\infty)>0$.

So if $F(0) > 0$, then there exists one root say $\hat{Y}_1$ between 0 and $\frac{A}{\alpha_1+d_1}$ and another root between $\frac{A}{\alpha_1+d_1}$ and ∞. Corresponding to this $\hat{Y}_1$, the values of $\hat{N}_1$, $\hat{Y}_2$, $\hat{N}_2$ and $\hat{E}$ are determined using (4.9) and (4.10) and we get a nontrivial equilibrium point $\bar{E}_3$ under the following condition

$$\frac{\beta_1\beta_2\frac{A}{d_1}\bar{\bar{N}}_2}{\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{\bar{N}}_2\}(\nu_1+\alpha_1+d_1)} = R_0' > 1.$$

Here R_0' is the threshold for the system (4.8).

4.2.2.1 Stability Analysis

Here, we present the stability analysis of the three equilibria and the local stability results of these equilibria are given in the following theorem.

THEOREM 4.4 The equilibrium $\bar{E}_1$ is unstable, the equilibrium $\bar{E}_2$ is stable if $R_0' < 1$, otherwise the equilibrium $\bar{E}_3$ exists and is locally asymptotically stable provided

$$\begin{vmatrix} c_4 & c_2 & c_0 \\ 1 & c_3 & c_1 \\ 0 & c_4 & c_2 \end{vmatrix} > 0, \quad \begin{vmatrix} c_4 & c_2 & c_0 & 0 \\ 1 & c_3 & c_2 & 0 \\ 0 & c_4 & c_2 & c_0 \\ 0 & 1 & c_3 & c_1 \end{vmatrix} > 0,$$

where c_0, c_1, c_2, c_3, and c_4 are given explicitly in the proof of the theorem.

Proof: Let M_i be the variational matrices corresponding to equilibrium points $\bar{E}_i$ for $i = 1, 2, 3$.

$$M_1 = \begin{pmatrix} -(\nu_1+\alpha_1+d_1) & 0 & \frac{\beta_1 A}{d_1} & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & 0 & 0 \\ 0 & 0 & -(\alpha_2+d_2) & 0 & 0 \\ 0 & 0 & 0 & r_2-\alpha_2+\delta_2 & \frac{Q_0+l\frac{A}{d_1}}{\delta_0} & 0 \\ 0 & l & 0 & 0 & -\delta_0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -(\nu_1+\alpha_1+d_1) & 0 & \frac{\beta_1 A}{d_1} & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & 0 & 0 \\ \beta_2\bar{\bar{N}}_2 & 0 & -\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{\bar{N}}_2\} & 0 & 0 \\ 0 & 0 & 0 & -\frac{r_2\bar{\bar{N}}_2}{K_2} & 0 \\ 0 & l & 0 & 0 & -\delta_0 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} -(\beta_1\,\hat{Y}_2+\nu_1+\alpha_1+d_1) & \beta_1\,\hat{Y}_2 & \beta_1(\hat{N}_1-\hat{Y}_1) & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & 0 & 0 \\ \beta_2\,(\hat{N}_2-\hat{Y}_2) & 0 & m_{33} & \beta_2\,\hat{Y}_1-(1-a')\frac{r_2}{K_2}\hat{Y}_2 & 0 \\ 0 & 0 & 0 & -\frac{r_2\,\hat{N}_2}{K_2} & \delta_2\,\hat{N}_2 \\ 0 & l & 0 & 0 & -\delta_0 \end{pmatrix},$$

where $m_{33} = -[\beta_2\,\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2]$.

From M_1 it is clear that $\bar{E}_1$ is unstable. From M_2 it is clear that $\bar{E}_2$ is also unstable under following condition,

$$\frac{\beta_1\beta_2\frac{A}{d_1}\bar{\bar{N}}_2}{\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{\bar{N}}_2\}(\nu_1+\alpha_1+d_1)} = R_0'(say) \;>\; 1.$$

The characteristic polynomial corresponding to the matrix M_3 is given by

$$\psi^5+c_4\,\psi^4+c_3\,\psi^3+c_2\,\psi^2+c_1\,\psi+c_0=0,$$

where

$$c_4 = \frac{r_2}{K_2}\hat{N}_2+\beta_1\,\hat{Y}_2+\delta_0+\nu_1+\alpha_1+2\,d_1+\beta_2\,\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2,$$

$$\begin{aligned} c_3 \;=\;& \beta_1\,\hat{Y}_2\,\alpha_1+\frac{r_2}{K_2}\,\hat{N}_2\left\{\delta_0+\beta_1\,\hat{Y}_2+\nu_1+\alpha_1+2d_1+\beta_2\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\\ &+\delta_0\left[\nu_1+\beta_1\,\hat{Y}_2+\alpha_1+2d_1+\beta_2\,\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right]\\ &+(d_1+\beta_2\,\hat{Y}_1)(\beta_1\,\hat{Y}_2+\nu_1+\alpha_1+d_1)\\ &+\left[\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right]\beta_1\,\hat{Y}_2+d_1\left[\beta_2\,\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right], \end{aligned}$$

$$\begin{aligned} c_2 \;=\;& \beta_1\,\hat{Y}_2\alpha_1\left[\delta_0+\beta_2\,\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2+\frac{r_2\,\hat{N}_2}{K_2}\right]\\ &+\frac{r_2\,\hat{N}_2}{K_2}\left\{\delta_0\left[\nu_1+\alpha_1+2d_1+\beta_2\,\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right]\right.\\ &+(d_1+\beta_2\,\hat{Y}_1)(\beta_1\,\hat{Y}_2+\nu_1+\alpha_1+d_1)\\ &\left.+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\beta_1\,\hat{Y}_2+d_1\left[\beta_2\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right]\right\}\\ &+\delta_0\left\{(d_1+\beta_2\,\hat{Y}_1)(\beta_1\,\hat{Y}_2+\nu_1+\alpha_1+d_1)+r_2\frac{\hat{N}_2}{K_2}\beta_1\hat{Y}_2\right.\\ &\left.+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\beta_1\,\hat{Y}_2+d_1\left[\beta_2\,\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right]\right\}\\ &+d_1\beta_2\,\hat{Y}_1(\beta_1\,\hat{Y}_2+\nu_1+\alpha_1+d_1)+d_1\,\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\beta_1\,\hat{Y}_2, \end{aligned}$$

$$\begin{aligned}
c_1 &= \frac{r_2}{K_2}\,\hat{N}_2\Big\{\delta_0(d_1+\beta_2\,\hat{Y}_1)(\beta_1\,\hat{Y}_2+\nu_1+\alpha_1+d_1)\\
&\quad +\delta_0\,\Big\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\Big\}\,\beta_1\,\hat{Y}_2+\delta_0\,d_1\Big[\beta_2\,\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\Big]\\
&\quad +d_1\,\beta_2\,\hat{Y}_1(\beta_1\,\hat{Y}_2+\nu_1+\alpha_1+d_1)+d_1\,\Big\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\Big\}\,\beta_1\,\hat{Y}_2\Big\}\\
&\quad +\delta_0\,d_1\,\beta_2\,\hat{Y}_1(\beta_1\,\hat{Y}_2+\nu_1+\alpha_1+d_1)+\delta_0\,d_1\,\Big\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\Big\}\,\beta_1\,\hat{Y}_2,\\
&\quad +\alpha_1\beta_1\hat{Y}_2\Big[\delta_0\frac{r_2\hat{N}_2}{K_2}+\delta_0\Big(\beta_2\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2\hat{N}_2}{K_2}\Big)\\
&\quad +\frac{r_2\hat{N}_2}{K_2}\Big(\beta_2\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2\hat{N}_2}{K_2}\Big)\Big]\\
c_0 &= \frac{r_2}{K_2}\,\hat{N}_2\delta_0\,d_1\,\beta_2\,\hat{Y}_1(\beta_1\,\hat{Y}_2+\nu_1+\alpha_1+d_1)+\frac{r_2}{K_2}\,\hat{N}_2\delta_0\,d_1\,\{\alpha_2+d_2\\
&\quad +(1-a')\frac{r_2}{K_2}\hat{N}_2\Big\}\,\beta_1\,\hat{Y}_2+\frac{r_2}{K_2}\,\hat{N}_2\,\delta_0\Big[\beta_2\,\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\Big]\,\beta_1\,\hat{Y}_2\alpha_1\\
&\quad +\delta_2\,\hat{N}_2\Big\{\beta_2\hat{Y}_1-(1-a')\frac{r_2}{K_2}\hat{Y}_2\Big\}\,\beta_1(\hat{N}_1-\hat{Y}_1)\alpha_1 l.
\end{aligned}$$

Then for the roots to have all real parts negative, it is necessary and sufficient that the following inequalities are satisfied,

$$c_4>0,\quad \begin{vmatrix} c_4 & c_2\\ 1 & c_3\end{vmatrix}>0,\quad \begin{vmatrix} c_4 & c_2 & c_0\\ 1 & c_3 & c_1\\ 0 & c_4 & c_2\end{vmatrix}>0,\quad \begin{vmatrix} c_4 & c_2 & c_0 & 0\\ 1 & c_3 & c_2 & 0\\ 0 & c_4 & c_2 & c_0\\ 0 & 1 & c_3 & c_1\end{vmatrix}>0,\quad \begin{vmatrix} c_4 & c_2 & c_0 & 0 & 0\\ 1 & c_3 & c_2 & 0 & 0\\ 0 & c_4 & c_2 & c_0 & 0\\ 0 & 1 & c_3 & c_1 & 0\\ 0 & 0 & c_4 & 0 & c_0\end{vmatrix}>0.$$

The first two conditions are always true. If the next two inequalities are satisfied then so is the fifth as $c_0>0$. Hence the equilibrium point $\bar{E}_3$ is locally asymptotically stable under the conditions mentioned in the theorem.

Nonlinear Analysis and Simulation: As in Case I, it is noted that the solution of system (4.8) is bounded by the solution of its corresponding system with $\alpha_1=0$, which is globally stable. Hence, as before, we conjecture that the equilibrium $\bar{E}_3$ of (4.8) may be globally stable in the interior of the region of attraction. To illustrate this global behaviour of nontrivial equilibrium point $\bar{E}_3$ and to show the effects of various parameters on the spread of malaria, we use simulation.

The system (4.8) is integrated using the fourth order Runge-Kutta method by using the same set of parameters as in Case I with $Q_0=Q_a$ and an additional parameter value $l=0.0005$, which satisfy the local stability conditions.

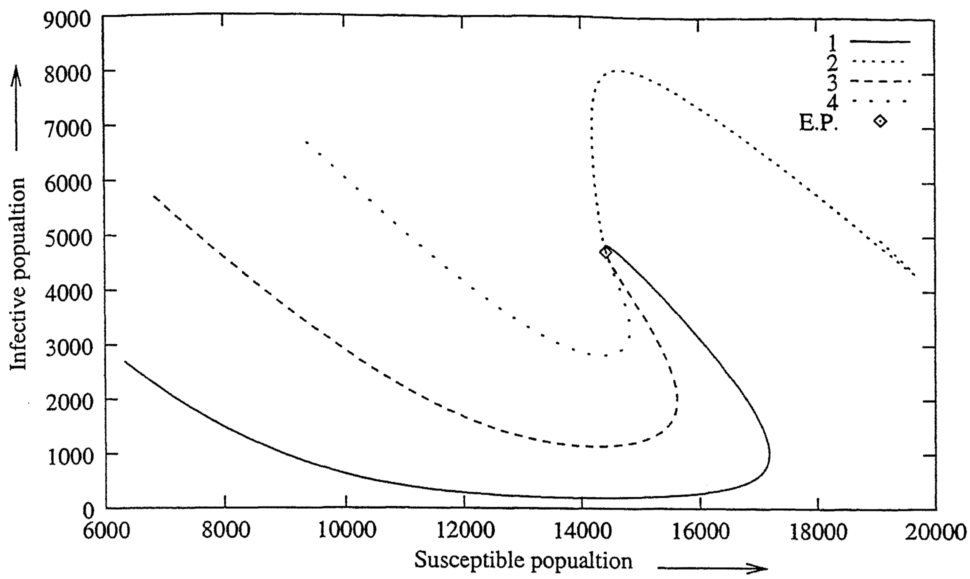


Figure 4.3: Variation of infective population Y_1 with susceptible population X_1.

The equilibrium $\bar{E}_3$ is calculated in this case as,

$\hat{Y}_1 = 4701.9,\ \hat{N}_1 = 19122.2,\ \hat{Y}_2 = 16825.0,\ \hat{N}_2 = 960912.2,\ \hat{E} = 29561.0.$

In this case also simulation is performed for different initial positions as shown in Fig. 4.3 implying the global stability of $\bar{E}_3$. Also in Figs. 4.4-4.7, we have shown the variation of Y_1 with time, for different r_2, δ_2, Q_0 and l respectively. We note that with the increase of any of these parameters, the infective population increases showing the effect of household discharges on the spread of malaria. The effect of immigration on the spread of malaria is the same as in Case I.

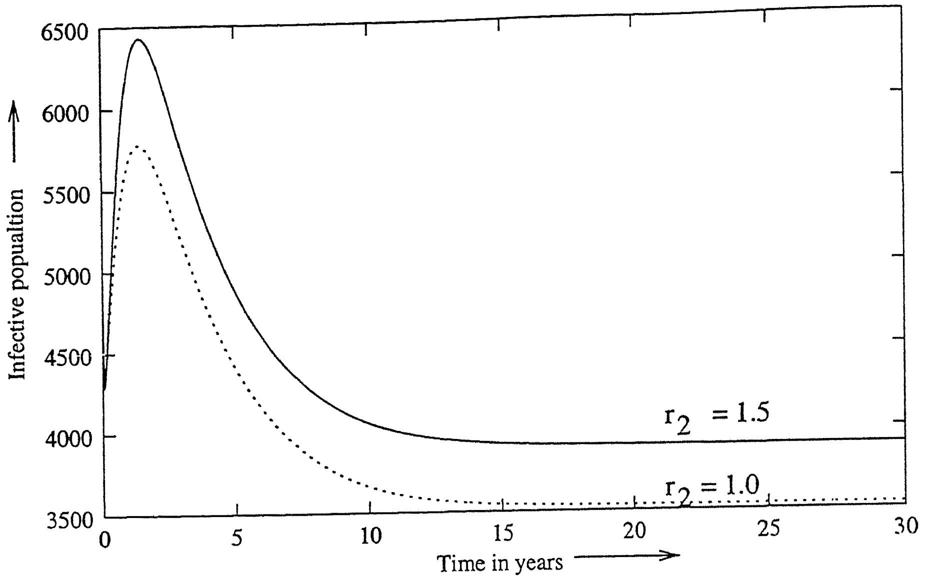


Figure 4.4: Variation of infective population Y_1 with time for different growth rate coefficients of the mosquito population.

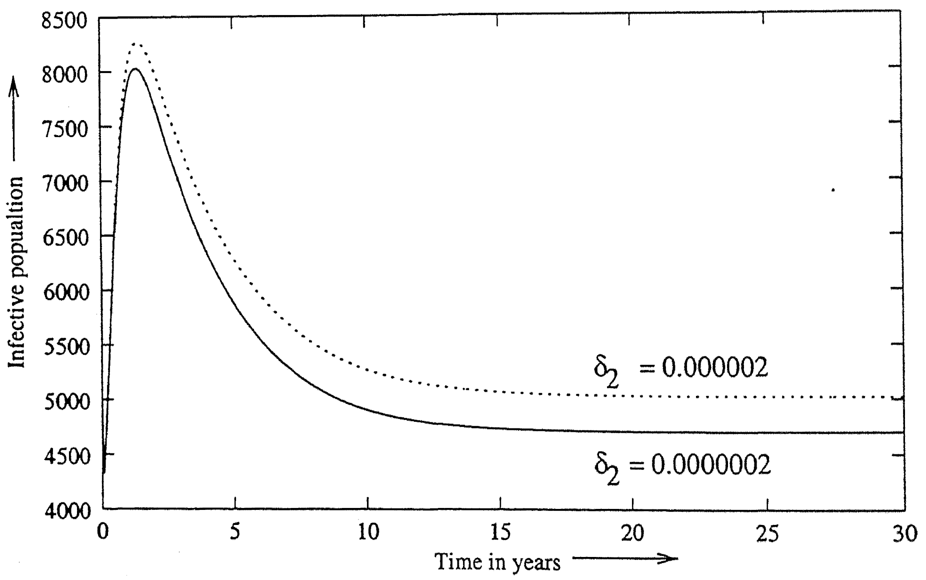


Figure 4.5: Variation of infective population Y_1 with time for different growth rate coefficients of the mosquito population due to environmental discharges.

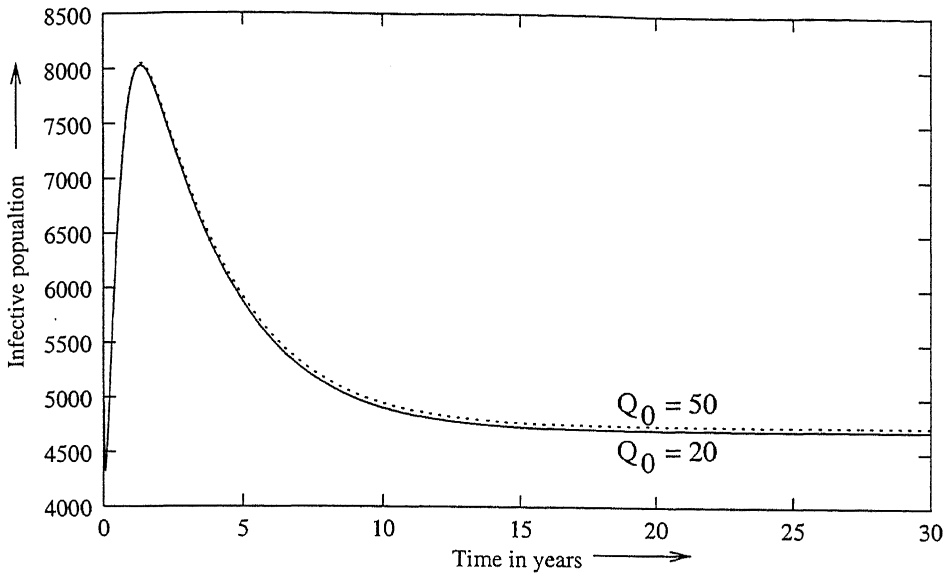


Figure 4.6: Variation of infective population Y_1 with time for different rates of cumulative environmental discharges.

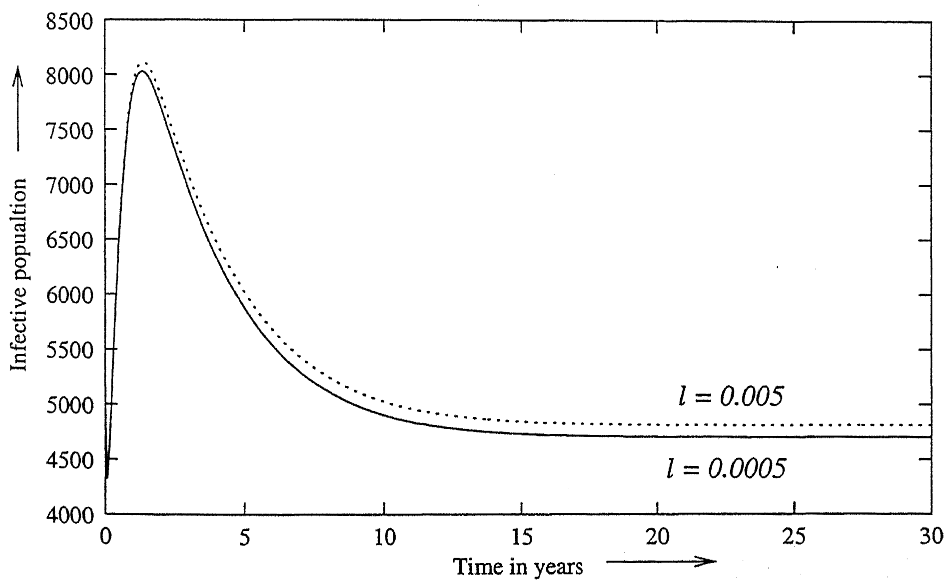


Figure 4.7: Variation of infective population Y_1 with time for different l.

4.3 SIS Model for Malaria with Logistic Growth of Human Population

Now let us consider an SIS model where the human population growth is logistic so that both the birth and death rates are density dependent in such a manner that the birth rate decreases and the death rate increases as the population density increases towards its carrying capacity (Gao and Hethcote 1992). As before the human population density is divided into susceptibles and infectives. Also the mosquito population is divided into susceptible mosquitoes and infective mosquitoes. It is assumed that mosquito population density grows logistically in the environment and its growth rate increases due to cumulative concentration of various discharges from household sources into the environment. Keeping these in mind, a mathematical model is proposed as follows:

$$
\begin{aligned}
\dot{X}_1 &= [b_1 - ar_1\frac{N_1}{K_1}]N_1 - \beta_1 X_1 Y_2 - [d_1 + (1-a)r_1\frac{N_1}{K_1}]X_1 + \nu_1 Y_1, \\
\dot{Y}_1 &= \beta_1 X_1 Y_2 - [\nu_1 + \alpha_1 + d_1 + (1-a)r_1\frac{N_1}{K_1}]Y_1, \\
\dot{N}_1 &= r_1[1 - \frac{N_1}{K_1}]N_1 - \alpha_1 Y_1, \\
\dot{X}_2 &= (b_2 - a'\frac{r_2}{K_2}N_2)N_2 - \{d_2 + (1-a')\frac{r_2}{K_2}N_2\}X_2 - \beta_2 X_2 Y_1 - \alpha_2 X_2 + \delta_2 N_2 E, \\
\dot{Y}_2 &= \beta_2 X_2 Y_1 - \{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2\}Y_2, \\
\dot{N}_2 &= r_2 N_2(1 - \frac{N_2}{K_2}) - \alpha_2 N_2 + \delta_2 N_2 E, \\
\dot{E} &= Q(N) - \delta_0 E, \quad 0 \le a \le 1, \quad 0 \le a' \le 1,
\end{aligned}
\tag{4.13}
$$

$$X_1(0) = X_{10} > 0,\ Y_1(0) = Y_{10} \ge 0,\ X_2(0) = X_{20} \ge 0, Y_2(0) = Y_{20} \ge 0,\ E(0) = E_0 > 0.$$

Here b_1 and d_1 are natural birth and death rates; $r_1 = b_1 - d_1$ is the growth rate constant and K_1 is the carrying capacity of the human population in the natural environment. All other parameters have already been defined in Section 4.2. For $0 < a < 1$, the birth rate decreases and the death rate increases as N_1 increases to its carrying capacity K_1. When $a = 1$, the model could be called simply logistic birth model as all of the restricted growth is due to a decreasing birth rate and the death rate is constant. Similarly, when $a = 0$, it could be called a logistic death model as all of the restricted growth is due to

an increasing death rate and the birth rate is constant.

In this section also, we analyze the following two cases:

(i) the rate of cumulative environmental discharges Q is constant,

(ii) Q is human population density dependent as described in Section 4.2.

4.3.1 Case I: $Q = Q_a$ a Constant

In this case the model (4.13) reduces to the following form,

$$\begin{aligned}
\dot{Y}_1 &= \beta_1(N_1 - Y_1)Y_2 - [\nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1 N_1}{K_1}]Y_1, \\
\dot{N}_1 &= r_1\left(1 - \frac{N_1}{K_1}\right)N_1 - \alpha_1 Y_1, \\
\dot{Y}_2 &= \beta_2(N_2 - Y_2)Y_1 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2\right\}Y_2, \\
\dot{N}_2 &= r_2\left(1 - \frac{N_2}{K_2}\right)N_2 - \alpha_2 N_2 + \delta_2 N_2 E, \\
\dot{E} &= Q_a - \delta_0 E.
\end{aligned} \tag{4.14}$$

Using the asymptotic values of E and N_2, $E = \frac{Q_a}{\delta_0} = \bar{E}$ (say) and $N_2 = \frac{K_2}{r_2}\{r_2 - \alpha_2 + \delta_2\frac{Q_a}{\delta_0}\} = \bar{N_2}$ (say) in the above set of equations, the behaviour of (4.14) can be given by the following subsystem,

$$\begin{aligned}
\dot{Y}_1 &= \beta_1(N_1 - Y_1)Y_2 - \left[\nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1 N_1}{K_1}\right]Y_1, \\
\dot{N}_1 &= r_1\left(1 - \frac{N_1}{K_1}\right)N_1 - \alpha_1 Y_1, \\
\dot{Y}_2 &= \beta_2(\bar{N_2} - Y_2)Y_1 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N_2}\right\}Y_2.
\end{aligned} \tag{4.15}$$

We note here that the region of attraction of the system (4.15) is given by,

$$T = \left\{(Y_1, N_1, Y_2) : 0 \le Y_1 \le N_1 \le K_1, 0 \le Y_2 \le \bar{N_2}\right\}.$$

The results of the equilibrium analysis is stated in the following theorem.

THEOREM 4.5 There exist the following three equilibria, namely (i) $E_1(0,0,0)$, *(ii)* $E_2(0, K_1, 0)$ *and (iii)* $E_3(\hat{Y}_1, \hat{N}_1, \hat{Y}_2)$. *The third equilibrium* E_3 *exists*

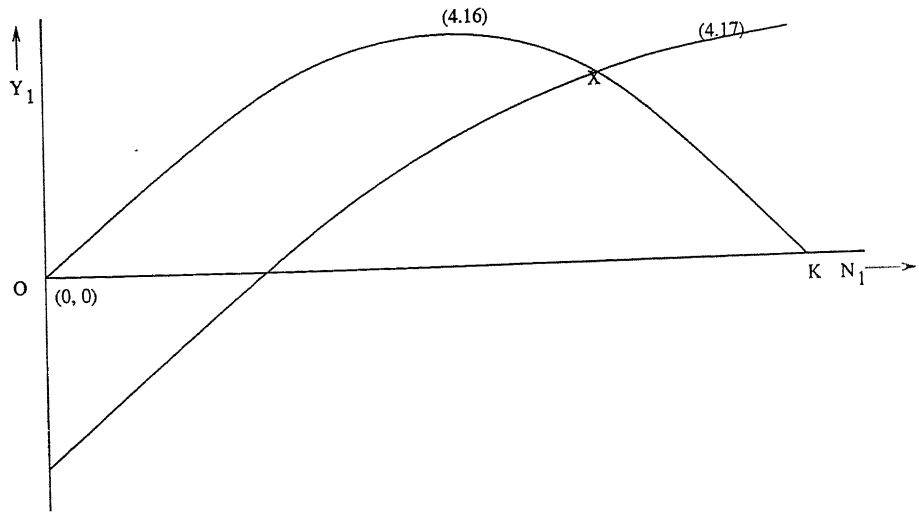


Figure 4.8: Existence of equilibrium point.

if

$$\frac{\beta_1\beta_2K_1\bar{N}_2}{\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\{\nu_1+\alpha_1+d_1+(1-a)r_1\}} = R_0' > 1.$$

Proof: The existence of E_1 or E_2 is trivial. The existence of the nontrivial equilibrium E_3 can be proved as follows by using the isocline method. From (4.15) we have,

$$Y_1 = \frac{r_1}{\alpha_1}\left(1-\frac{N_1}{K_1}\right)N_1 \tag{4.16}$$

and

$$Y_1 = \frac{\beta_1\beta_2N_1\bar{N}_2 - \left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\left\{\nu_1+\alpha_1+d_1+(1-a)\frac{r_1N_1}{K_1}\right\}}{\beta_1\beta_2\bar{N}_2+\beta_2\left\{\nu_1+\alpha_1+d_1+(1-a)\frac{r_1N_1}{K_1}\right\}}. \tag{4.17}$$

Clearly (4.16) is a parabola passing through (0, 0) and $(K_1, 0)$ with vertex at $(\frac{K_1}{2}, \frac{r_1K_1}{4\alpha_1})$. From equation (4.17), we have the following:

$$\text{for } N_1 = 0,\ Y_1 = \frac{-\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\}(\nu_1+\alpha_1+d_1)}{\beta_1\beta_2\bar{N}_2+\beta_2(\nu_1+\alpha_1+d_1)} < 0,$$

$$\text{for } Y_1 = 0,\ N_1 = \frac{(\nu_1+\alpha_1+d_1)\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\}}{\beta_1\beta_2\bar{N}_2-\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\}(1-a)\frac{r_1}{K_1}} > 0, \text{ for } R_0' > 1,$$

$$\text{for } N_1 = K_1,\ Y_1 = \frac{\beta_1\beta_2 K_1 \bar{N}_2 - \{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\}\{\nu_1 + \alpha_1 + d_1) + (1-a)r_1\}}{\beta_1\beta_2 N_2 + \beta_2\{\nu_1 + \alpha_1 + d_1 + (1-a)r_1\}},$$

which is positive provided

$$\frac{\beta_1\beta_2 K_1 \bar{N}_2}{\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\}\{\nu_1 + \alpha_1 + d_1 + (1-a)r_1\}} = R_0' > 1.$$

Also the slope of (4.17) is given by

$$\frac{dY_1}{dN_1} = \frac{\beta_1\beta_2\bar{N}_2 - \{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\}(1-a)\frac{r_1}{K_1} - \beta_2(1-a)\frac{r_1}{K_1}Y_1}{\beta_1\beta_2\bar{N}_2 + \beta_2(\nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1}{K_1}N_1)},$$

which is positive using (4.17) and $R_0' > 1$ for $N_1 > 0$. Thus, plotting (4.16) and (4.17) in Fig. 4.8, it is clear that there exists a unique nontrivial equilibrium point $E_3(\hat{Y}_1, \hat{N}_1, \hat{Y}_2)$. Here R_0' is the threshold for the system (4.15), which we will also see from the instability condition of the disease free equilibrium point.

4.3.1.1 Stability Analysis

Now we present the stability analysis of these equilibria. We state the local stability results of these three equilibria in the following theorem.

THEOREM 4.6 The equilibrium E_1 is unstable, the equilibrium E_2 is stable if $R_0' < 1$ and unstable if $R_0' > 1$ in which case the equilibrium E_3 exists. The equilibrium E_3, if it exists, is locally asymptotically stable when $a_3 > 0$ and $a_1a_2 - a_3 > 0$, where a_1, a_2 and a_3 are given in the proof of the theorem.

Proof: The variational matrix at (Y_1, N_1, Y_2) corresponding to the system (4.15) is

$$M = \begin{pmatrix} -(\beta_1 Y_2 + \nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1 N_1}{K_1}) & \beta_1 Y_2 & \beta_1(N_1 - Y_1) \\ -\alpha_1 & r_1 - \frac{2r_1N_1}{K_1} & 0 \\ \beta_2(\bar{N}_2 - Y_2) & 0 & m_{33} \end{pmatrix},$$

where $m_{33} = -[\beta_2 Y_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2]$.

At the equilibrium point $E_1(0,0,0)$, the variational matrix M_1 is given by

$$M_1 = \begin{pmatrix} -(\nu_1+\alpha_1+d_1) & 0 & 0 \\ -\alpha_1 & r_1 & 0 \\ \beta_2\bar{N}_2 & 0 & -\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\} \end{pmatrix}.$$

From M_1, it is clear that it has one positive characteristic root r_1 implying instability of this equilibrium point.

At equilibrium point $E_2(0, K_1, 0)$ the variational matrix M_2 is given by

$$M_2 = \begin{pmatrix} -\{\nu_1+\alpha_1+d_1+(1-a)r_1\} & 0 & \beta_1 K_1 \\ -\alpha_1 & -r_1 & 0 \\ \beta_2\bar{N}_2 & 0 & -\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\} \end{pmatrix}.$$

Clearly one characteristic root of the above matrix is $-r_1$ and other two roots are

$$\psi^2 + \{\nu_1+\alpha_1+d_1+(1-a)r_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\}\psi$$
$$+\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\}\{\nu_1+\alpha_1+d_1+(1-a)r_1\} - \beta_1\beta_2 K_1\bar{N}_2 = 0.$$

Using the Routh-Hurwitz criteria, we get the condition for stability of the equilibrium E_2 as $R_0' < 1$, otherwise this equilibrium becomes unstable and the equilibrium point E_3 exists.

At equilibrium point E_3 the variational matrix M_3 is given by

$$M_3 = \begin{pmatrix} -(\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+\overline{1-a}\,\frac{r_1\hat{N}_1}{K_1}) & \beta_1\hat{Y}_2 & \beta_1(\hat{N}_1-\hat{Y}_1) \\ -\alpha_1 & r_1-\frac{2r_1}{K_1}\hat{N}_1 & 0 \\ \beta_2(\bar{N}_2-\hat{Y}_2) & 0 & m_{33} \end{pmatrix},$$

where $m_{33} = -[\beta_2\hat{Y}_1 + \{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\}]$.

The characteristic polynomial corresponding to the above matrix is given by

$$\psi^3 + a_1\psi^2 + a_2\psi + a_3 = 0,$$

where

$$\begin{aligned} a_1 &= \frac{r_1}{K_1}(2\hat{N}_1 - K_1) + \left\{\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+(1-a)\frac{r_1\hat{N}_1}{K_1}\right\} \\ &\quad + \left\{\beta_2\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\right\}, \end{aligned}$$

$$\begin{aligned}
a_2 &= \frac{r_1}{K_1}(2\hat{N}_1 - K_1)\left\{\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + (1-a)\ \frac{r_1\hat{N}_1}{K_1} + \beta_2\hat{Y}_1 + \alpha_2 + d_2\right. \\
&\left. +(1-a')\frac{r_2}{K_2}\bar{N}_2\right\} + \left\{\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1\hat{N}_1}{K_1}\right\} \\
&\times\left[\beta_2\hat{Y}_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right] + \alpha_1\beta_1\hat{Y}_2 + \beta_1\beta_2(\hat{N}_1 - \hat{Y}_1)(\bar{N}_2 - \hat{Y}_2), \\
a_3 &= \frac{r_1}{K_1}\left(2\hat{N}_1 - K_1\right)\beta_2\hat{Y}_1\left\{\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1\hat{N}_1}{K_1}\right\} \\
&+\frac{r_1}{K_1}\left(2\hat{N}_1 - K_1\right)\left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\beta_1\hat{Y}_2 \\
&+\alpha_1\beta_1\hat{Y}_2\left[\beta_2\hat{Y}_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\right].
\end{aligned}$$

It may be noted here that $a_1 > 0$, hence from the Routh-Hurwitz criteria, E_3 is locally asymptotically stable if $a_3 > 0$ and $a_1a_2 - a_3 > 0$.

Remark: It is noted that for $\hat{N}_1 > \frac{K_1}{2}$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$. Hence E_3 is locally asymptotically stable under the assumption $\hat{N}_1 > \frac{K_1}{2}$.

Nonlinear Analysis and Simulation:

We first note that the system (4.15) is globally stable when the disease related death rate α_1 is zero. Since system (4.15) is bounded by its corresponding system (4.15) with $\alpha_1 = 0$, using a comparison theorem (Lakshmikantham and Leela 1969), it is concluded that solution of (4.15) is bounded by the solution of (4.15) with $\alpha_1 = 0$. Thus, we conjecture that the system (4.15) is globally stable in the interior of the region of attraction. To support this result, system (4.15) is integrated by the fourth order Runge-Kutta method using the following set of parameters in the simulation, which satisfy the local stability condition.

$$\beta_1 = 0.00000022 = \beta_2,\quad \nu_1 = 0.012,\quad a = 0.3,\quad \alpha_1 = 0.0005,\quad d_1 = 0.0004,$$

$$r_1 = 0.0003,\ K_1 = 50000,\ \alpha_2 = 0.045\ ,\ r_2 = 1,\ a' = 0.999,$$

$$d_2 = 0.02,\ \delta_2 = 0.0000002,\ Q_a = 20,\ \delta_0 = 0.001,\ K_2 = 1000000.$$

The equilibrium values for this set of parameters are given by

$$\hat{Y}_1 = 7475.6392,\quad \hat{N}_1 = 26424.8209,\quad \hat{Y}_2 = 23331.6445.$$

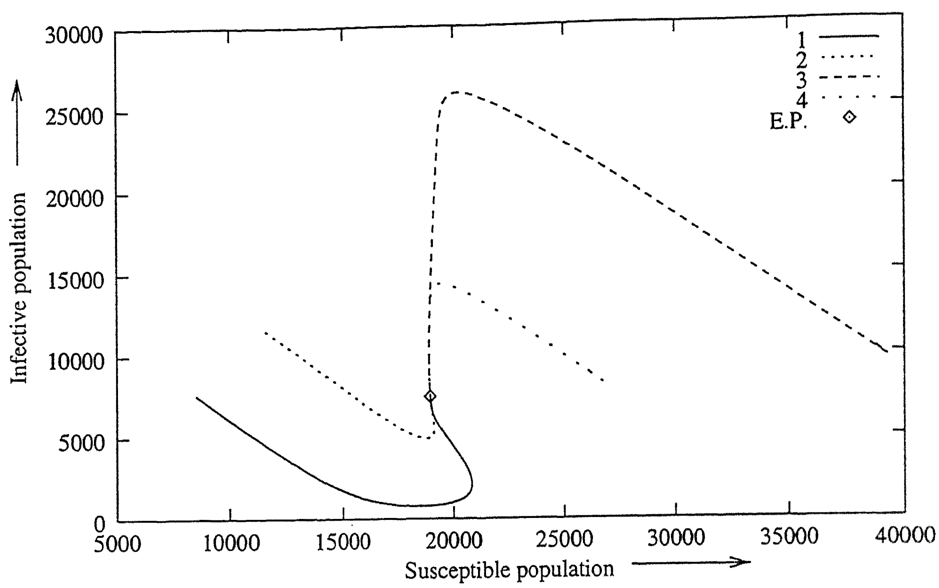


Figure 4.9: Variation of infective population with susceptible population.

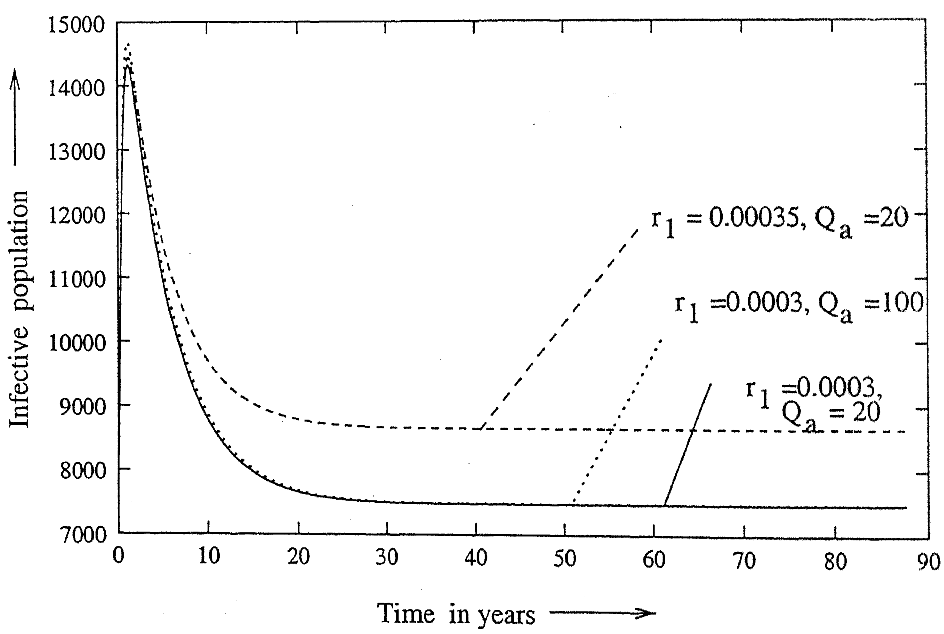


Figure 4.10: Variation of infective population with time for different growth rates of human population.

In Fig. 4.9, we have plotted the infective population against the susceptible population for different initial positions 1, 2, 3 and 4 and from the solution curves, we infer that system is globally stable about the endemic equilibrium point $(\hat{Y}_1,\ \hat{N}_1,\ \hat{Y}_2,)$, provided that we start away from E_1 and E_2. From Fig. 4.10, we note that when the growth rate of human population r_1 increases, the endemic infective human population increases. Also when Q_a increases, the endemic infective population increases.

4.3.2 Case II: $Q = Q_0 + lN_1$

In this case, since $X_1 + Y_1 = N_1$ and $X_2 + Y_2 = N_2$, the system (4.13) reduces to the following form,

$$\begin{aligned}
\dot{Y}_1 &= \beta_1(N_1 - Y_1)Y_2 - \left[\nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1N_1}{K_1}\right]Y_1,\\
\dot{N}_1 &= r_1N_1\left(1 - \frac{N_1}{K_1}\right) - \alpha_1Y_1,\\
\dot{Y}_2 &= \beta_2(N_2 - Y_2)Y_1 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2\right\}Y_2,\\
\dot{N}_2 &= r_2N_2\left(1 - \frac{N_2}{K_2}\right) - \alpha_2N_2 + \delta_2N_2E,\\
\dot{E} &= Q_0 + lN_1 - \delta_0E.
\end{aligned} \tag{4.18}$$

In this case the region of attraction is given by,

$$T_1 = \left\{(Y_1, N_1, Y_2, N_2, E) : 0 \le Y_1 \le N_1 \le K_1, 0 \le Y_2 \le N_2 \le K_2, 0 < E \le \frac{Q_0 + lK_1}{\delta_0}\right\}.$$

The results of an equilibrium analysis is stated in the following theorem.

THEOREM 4.7 There exist the following five equilibria, namely
(i) $E_1\left(0,0,0,0,\frac{Q_0}{\delta_0}\right)$, *(ii)* $E_2\left(0,K_1,0,0,\frac{Q_0+lK_1}{\delta_0}\right)$,
(iii) $E_3\left(0,0,0,\bar{N}_2,\frac{Q_0}{\delta_0}\right)$, *where* $\bar{N}_2 = \frac{K_2}{r_2}\left\{r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0}\right\}$,
(iv) $E_4\left(0,K_1,0,N_2^*,\frac{Q_0+lK_1}{\delta_0}\right)$, *where* $N_2^* = \frac{K_2}{r_2}\left\{r_2 - \alpha_2 + \delta_2\frac{Q_0+lK_1}{\delta_0}\right\}$
and (v) $E_5(\hat{Y}_1, \hat{N}_1, \hat{Y}_2, \hat{N}_2, \hat{E})$. E_5 *exists provided*

$$\frac{\dfrac{\beta_1\beta_2K_1K_2}{r_2}\left\{r_2 - \alpha_2 + \delta_2\dfrac{Q_0 + lK_1}{\delta_0}\right\}}{\left\{\alpha_2 + d_2 + (1-a')\dfrac{r_2}{K_2}N_2^*\right\}\{\nu_1 + \alpha_1 + d_1 + (1-a)r_1\}} = R'_{00}\ \textit{(say)} > 1.$$

For uniqueness of this equilibrium we need an additional sufficient condition as $\frac{dY_1}{dN_1} > 0$*, for all* $N_1 > 0$*, where* Y_1 *is given by*

$$Y_1 = \frac{\beta_1\beta_2 N_1\ N_2 - \left[\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\ N_2\right]\left\{\nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1 N_1}{K_1}\right\}}{\left[\beta_2\left\{\nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1 N_1}{K_1}\right\} + \beta_1\beta_2 N_2\right]},$$

with $N_2 = \frac{K_2}{r_2}\left\{r_2 - \alpha_2 + \delta_2\frac{Q_0 + lN_1}{\delta_0}\right\}$.

Proof: The existence of E_1, E_2, E_3 or E_4 is obvious. We prove here the existence of E_5. Setting right hand side of (4.18) to zero, we get the following equations, when $N_1 \neq 0$, $N_1 \neq K_1$, $N_2 \neq 0$, $N_2 \neq \bar{N}_2$, $N_2 \neq N_2^*$,

$$E = \frac{Q_0 + lN_1}{\delta_0}, \quad N_2 = \frac{K_2}{r_2}\{r_2 - \alpha_2 + \delta_2\frac{Q_0 + lN_1}{\delta_0}\}, \tag{A1}$$

$$Y_2 = \frac{\left[\nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1}{K_1}N_1\right] Y_1}{\beta_1(N_1 - Y_1)}, \tag{A2}$$

$$Y_1 = \frac{r_1}{\alpha_1}(1 - \frac{N_1}{K_1})N_1, \tag{4.19}$$

$$Y_1 = \frac{\beta_1\beta_2 N_1\ N_2 - [\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\ N_2]\{\nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1 N_1}{K_1}\}}{[\beta_2\{\nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1 N_1}{K_1}\} + \beta_1\beta_2 N_2]}, \tag{4.20}$$

where N_2 is given in equation (A1).

Now we use the isocline method for finding the equilibrium point E_5 from (4.19) and (4.20). Clearly (4.19) is a parabola in the $N_1 - Y_1$ plane, passing through $(0,0)$ and $(K_1, 0)$ with vertex $\left(\frac{K_1}{2}, \frac{r_1 K_1}{4\alpha_1}\right)$.

(i) From (4.20), we get when $N_1 = 0$,

$$Y_1 = \frac{-\left[\alpha_2 + d_2 + (1-a')\left\{r_2 - \alpha_2 + \frac{\delta_2}{\delta_0}Q_0\right\}\right](\nu_1 + \alpha_1 + d_1)}{\beta_2(\nu_1 + \alpha_1 + d_1) + \beta_1\beta_2\frac{K_2}{r_2}\left\{r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0}\right\}} < 0.$$

(ii) Also when $N_1 = K_1$, equation (4.20) gives

$$Y_1 = \frac{\beta_1\beta_2 N_2^* K_1 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\right\}\left\{\nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1}{K_1}K_1\right\}}{\beta_2\left[\beta_1 N_2^* + \{\nu_1 + \alpha_1 + d_1 + (1-a)r_1\}\right]} > 0,$$

provided

$$\frac{\frac{\beta_1\beta_2 K_1 K_2}{r_2}\left\{r_2 - \alpha_2 + \delta_2\frac{Q_0 + lK_1}{\delta_0}\right\}}{\left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\right\}\{\nu_1 + \alpha_1 + d_1 + (1-a)r_1\}} = R'_{00} > 1, \tag{4.21}$$

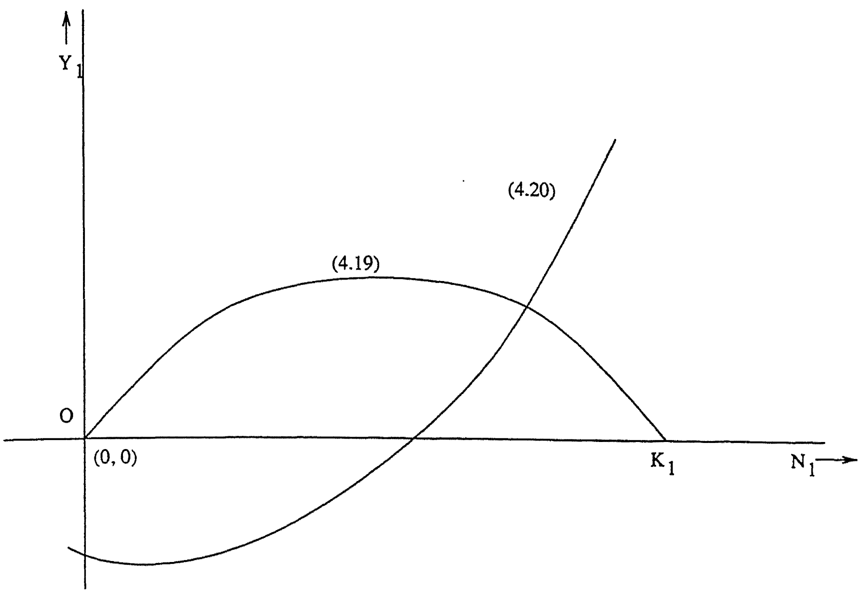


Figure 4.11: Existence of equilibrium point.

where R'_{00} is the threshold of the system (4.18). Hence (4.20) intersects the N_1 axis at least once.

(iii) For $Y_1 = 0$, we get the following quadratic

$$\left\{\beta_1\beta_2\frac{K_2}{r_2} - (1-a)(1-a')\frac{r_1}{K_1}\right\}\frac{\delta_2 l}{\delta_0}N_1^2 + \left[\beta_1\beta_2\frac{K_2}{r_2}\left\{r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0}\right\}\right.$$
$$-(1-a)\frac{r_1}{K_1}\left\{\alpha_2 + d_2 + (1-a')(r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0})\right\} - (1-a')(\nu_1 + \alpha_1 + d_1)\left.\frac{\delta_2 l}{\delta_0}\right] N_1$$
$$-(\nu_1 + \alpha_1 + d_1)\left\{\alpha_2 + d_2 + (1-a')(r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0})\right\} = 0. \tag{4.22}$$

It is noted that under condition (4.21), the coefficient of N_1^2 is positive and the constant term in (4.22) is negative which implies it has one positive and one negative root. Also, if $\frac{dY_1}{dN_1} > 0$ is positive, then from Fig. 4.11, it is clear that there exists a unique intersection point $(\hat{Y}_1, \hat{N}_1)$ satisfying both (4.19) and (4.20) which corresponds to the fifth equilibrium point E_5 under condition (4.21) and $\frac{dY_1}{dN_1} > 0$. E_5 is completely determined using (A1) and (A2).

4.3.2.1 Stability Analysis

We now present the stability analysis of these equilibria. The local stability results of these equilibria are stated in the following theorem.

THEOREM 4.8 The three equilibria E_1, E_2 and E_3 are unstable. The equilibrium E_4 is stable if $R'_{00} < 1$. If $R'_{00} < 1$ it is unstable and the equilibrium E_5 exists. The fifth equilibrium E_5, if it exists, is locally asymptotically stable provided

$$a_0 > 0, \quad \begin{vmatrix} a_4 & a_2 \\ 1 & a_3 \end{vmatrix} > 0, \quad \begin{vmatrix} a_4 & a_2 & a_0 \\ 1 & a_3 & a_1 \\ 0 & a_4 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_4 & a_2 & a_0 & 0 \\ 1 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_0 \\ 0 & 1 & a_3 & a_1 \end{vmatrix} > 0,$$

where a_0, a_1, a_2, a_3 and a_4 are given explicitly in the proof of the theorem.

Proof: The variational matrix M corresponding to the system (4.18) is given by

$$M = \begin{pmatrix} m_{11} & \beta_1 Y_2 - (1-a)\frac{r_1}{K_1}Y_1 & \beta_1(N_1 - Y_1) & 0 & 0 \\ -\alpha_1 & r_1 - \frac{2r_1N_1}{K_1} & 0 & 0 & 0 \\ \beta_2(N_2 - Y_2) & 0 & m_{33} & \beta_2 Y_1 - (1-a')\frac{r_2}{K_2}Y_2 & 0 \\ 0 & 0 & 0 & m_{44} & \delta_2 N_2 \\ 0 & l & 0 & 0 & -\delta_0 \end{pmatrix},$$

where $m_{11} = -\left\{\beta_1 Y_2 + \nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1 N_1}{K_1}\right\}$,
$m_{33} = -\left[\beta_2 Y_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2\right]$ and $m_{44} = r_2 - \alpha_2 - \frac{2r_2N_2}{K_2} + \delta_2 E$.
The variational matrix M_1 at equilibrium point E_1 is given by

$$M_1 = \begin{pmatrix} -(\nu_1 + \alpha_1 + d_1) & 0 & 0 & 0 & 0 \\ -\alpha & r_1 & 0 & 0 & 0 \\ 0 & 0 & -(\alpha_2 + d_2) & 0 & 0 \\ 0 & 0 & 0 & r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0} & 0 \\ 0 & l & 0 & 0 & -\delta_0 \end{pmatrix}.$$

Here two characteristic roots of M_1 are positive. Hence this equilibrium point is unstable.
The variational matrix M_2 at equilibrium point E_2 is given by

$$M_2 = \begin{pmatrix} -(\nu_1 + \alpha_1 + d_1 + (1-a)r_1) & 0 & \beta_1 K_1 & 0 & 0 \\ -\alpha & -r_1 & 0 & 0 & 0 \\ 0 & 0 & -(\alpha_2 + d_2) & 0 & 0 \\ 0 & 0 & 0 & r_2 - \alpha_2 + \delta_2\frac{Q_0 + lK_1}{\delta_0} & 0 \\ 0 & l & 0 & 0 & -\delta_0 \end{pmatrix}.$$

Here one characteristic root, $\left(r_2 - \alpha_2 + \delta_2 \frac{Q_0 + lK_1}{\delta_0}\right)$, is positive implying that this equilibrium point is unstable.

The variational matrix M_3 at equilibrium point E_3 is given by

$$M_3 = \begin{pmatrix} -(\nu_1 + \alpha_1 + d_1) & 0 & 0 & 0 & 0 \\ -\alpha_1 & r_1 & 0 & 0 & 0 \\ \beta_2 \bar{N}_2 & 0 & -\left[\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right] & 0 & 0 \\ 0 & 0 & 0 & -\frac{r_2}{K_2}\bar{N}_2 & \delta_2 \bar{N}_2 \\ 0 & l & 0 & 0 & -\delta_0 \end{pmatrix}.$$

Here also one characteristic root r_1 is positive implying instability of this equilibrium point.

The variational matrix M_4 at E_4 is given by

$$M_4 = \begin{pmatrix} -\{\nu_1 + \alpha_1 + d_1 + (1-a)r_1\} & 0 & \beta_1 K_1 & 0 & 0 \\ -\alpha_1 & -r_1 & 0 & 0 & 0 \\ \beta_2 N_2^* & 0 & -\left[\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\right] & 0 & 0 \\ 0 & 0 & 0 & -\frac{r_2}{K_2}N_2^* & \delta_2 N_2^* \\ 0 & l & 0 & 0 & -\delta_0 \end{pmatrix}.$$

Here the characteristic polynomial is given by

$$(r_1 + \psi)\left(r_2 - \alpha_2 + \delta_2 \frac{(Q_0 + lK_1)}{\delta_0} + \psi\right)(\delta_0 + \psi)\left\{(\nu_1 + \alpha_1 + d_1 + (1-a)\, r_1 + \psi)\right.$$

$$\left. \times \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^* + \psi\right\} - \beta_1\beta_2 \frac{K_1 K_2}{r_2}\left(r_2 - \alpha_2 + \delta_2\, \frac{Q_0 + lK_1}{\delta_0}\right)\right\} = 0.$$

Clearly three roots are negative and other roots are given by the following quadratic

$$\psi^2 + \left\{\nu_1 + \alpha_1 + d_1 + (1-a)\, r_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\right\}\psi$$

$$+\{\nu_1 + \alpha_1 + d_1 + (1-a)\, r_1\}\left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\right\} - \beta_1\beta_2\frac{K_1 K_2}{r_2}\left(r_2 - \alpha_2 + \delta_2 \frac{Q_0 + lK_1}{\delta_0}\right) = 0.$$

Hence this equilibrium point will be stable if

$$(\nu_1 + \alpha_1 + d_1 + (1-a)\, r_1)\left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\right\} > \beta_1\beta_2\frac{K_1 K_2}{r_2}\left(r_2 - \alpha_2 + \delta_2 \frac{Q_0 + lK_1}{\delta_0}\right),$$

i.e. $R'_{00} < 1$, otherwise if $R'_{00} > 1$ it is unstable and the fifth equilibrium point will exist.

The variational matrix M_5 at equilibrium point E_5 is given by

$$M_5 = \begin{pmatrix} m_{11} & \beta_1 \hat{Y}_2 - \overline{1-a}\,\frac{r_1}{K_1}\hat{Y}_1 & \beta_1(\hat{N}_1 - \hat{Y}_1) & 0 & 0 \\ -\alpha_1 & r_1 - \frac{2r_1}{K_1}\hat{N}_1 & 0 & 0 & 0 \\ \beta_2(\hat{N}_2 - \hat{Y}_2) & 0 & m_{33} & \beta_2 \hat{Y}_1 - (1-a')\frac{r_2}{K_2}\hat{Y}_2 & 0 \\ 0 & 0 & 0 & -\frac{r_2}{K_2}\hat{N}_2 & \delta_2\, \hat{N}_2 \\ 0 & l & 0 & 0 & -\delta_0 \end{pmatrix},$$

$m_{11} = -\{\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + (1-a)\, r_1\frac{\hat{N}_1}{K_1}\}$ and $m_{33} = -[\beta_2\hat{Y}_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\hat{N}_2]$

In this case, the characteristic polynomial is given by

$$\psi^5 + a_4\psi^4 + a_3\psi^3 + a_2\psi^2 + a_1\psi + a_0 = 0,$$

where

$$\begin{aligned}
a_4 &= \beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1}{K_1}\hat{N}_1 + \frac{2r_1}{K_1}\hat{N}_1 - r_1 + \beta_2\hat{Y}_1 \\
&\quad + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\hat{N}_2 + \frac{r_2}{K_2}\hat{N}_2 + \delta_0 > 0, \\
a_3 &= \left\{\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1}{K_1}\hat{N}_1\right\}\left\{\frac{2r_1}{K_1}(\hat{N}_1 - \frac{K_1}{2}) + \beta_2\hat{Y}_1\right. \\
&\quad \left. + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\hat{N}_2 + \frac{r_2}{K_2}\hat{N}_2 + \delta_0\right\} \\
&\quad + \frac{2r_1}{K_1}(\hat{N}_1 - \frac{K_1}{2})\left\{\beta_2\hat{Y}_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\hat{N}_2 + \frac{r_2}{K_2}\hat{N}_2 + \delta_0\right\} \\
&\quad + \left[\beta_2 Y_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\hat{N}_2\right]\left(\frac{r_2}{K_2}\hat{N}_2 + \delta_0\right) + \frac{r_2}{K_2}\hat{N}_2\delta_0 \\
&\quad + \alpha_1\left\{\beta_1\hat{Y}_2 - (1-a)\frac{r_1}{K_1}\hat{Y}_1\right\} - \beta_1\beta_2(\hat{N}_1 - \hat{Y}_1)(\hat{N}_2 - \hat{Y}_2), \\
a_2 &= \left\{\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1}{K_1}\hat{N}_1\right\}\frac{2r_1}{K_1}\left(\hat{N}_1 - \frac{K_1}{2}\right)\left\{\beta_2\hat{Y}_1 + \alpha_2 + d_2\right. \\
&\quad \left. + (1-a')\frac{r_2}{K_2}\hat{N}_2 + \frac{r_2}{K_2}\hat{N}_2 + \delta_0\right\} \\
&\quad + \frac{2r_1}{K_1}\left(\hat{N}_1 - \frac{K_1}{2}\right)\left[\beta_2\hat{Y}_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\hat{N}_2\right]\left(\frac{r_2}{K_2}\hat{N}_2 + \delta_0\right) \\
&\quad + \left[\beta_2\hat{Y}_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\hat{N}_2\right] \\
&\quad \times \frac{r_2}{K_2}\hat{N}_2\left\{\delta_0 + \beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1}{K_1}\hat{N}_1\right\} \\
&\quad + \frac{r_2}{K_2}\hat{N}_2\delta_0\left\{\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1}{K_1}\hat{N}_1 + \frac{2r_1}{K_1}\left(\hat{N}_1 - \frac{K_1}{2}\right)\right\} \\
&\quad + \delta_0\left\{\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1}{K_1}\hat{N}_1\right\}\left[\beta_2\hat{Y}_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\hat{N}_2\right] \\
&\quad + \alpha_1\left\{\beta_1\hat{Y}_2 - (1-a)\frac{r_1}{K_1}\hat{Y}_1\right\}\left\{\beta_2\hat{Y}_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\hat{N}_2 + \frac{r_2}{K_2}\hat{N}_2 + \delta_0\right\} \\
&\quad - \beta_1\beta_2(\hat{N}_1 - \hat{Y}_1)(\hat{N}_2 - \hat{Y}_2)\left\{\frac{2r_1}{K_1}\left(\hat{N}_1 - \frac{K_1}{2}\right) + \frac{r_2}{K_2}\hat{N}_2 + \delta_0\right\}, \\
a_1 &= \left\{\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + (1-a)\frac{r_1}{K_1}\hat{N}_1\right\}\frac{2r_1}{K_1}\left(\hat{N}_1 - \frac{K_1}{2}\right) \\
&\quad \times \left[\beta_2\hat{Y}_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\hat{N}_2\right]\left(\frac{r_2}{K_2}\hat{N}_2 + \delta_0\right) \\
&\quad + \left[\beta_2\hat{Y}_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\hat{N}_2\right]
\end{aligned}$$

$$
\begin{aligned}
& \times\frac{r_2}{K_2}\hat{N}_2\delta_0\left\{\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+(1-a)\frac{r_1}{K_1}\hat{N}_1+\frac{2r_1}{K_1}\left(\hat{N}_1-\frac{K_1}{2}\right)\right\} \\
& +\frac{r_2}{K_2}\hat{N}_2\delta_0\left\{\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+(1-a)\frac{r_1}{K_1}\hat{N}_1\right\}\frac{2r_1}{K_1}\left(\hat{N}_1-\frac{K_1}{2}\right) \\
& +\alpha_1\left\{\beta_1\hat{Y}_2-(1-a)\frac{r_1}{K_1}\hat{Y}_1\right\} \\
& \times\left[\left\{\beta_2\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\left(\frac{r_2}{K_2}\hat{N}_2+\delta_0\right)+\frac{r_2}{K_2}\hat{N}_2\delta_0\right] \\
& -\beta_1\beta_2(\hat{N}_1-\hat{Y}_1)(\hat{N}_2-\hat{Y}_2)\left\{\frac{2r_1}{K_1}\left(\hat{N}_1-\frac{K_1}{2}\right)\left(\frac{r_2}{K_2}\hat{N}_2+\delta_0\right)+\frac{r_2}{K_2}\hat{N}_2\delta_0\right\}, \\
a_0 = & \left\{\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+(1-a)\frac{r_1}{K_1}\hat{N}_1\right\}\frac{2r_1}{K_1}\left(\hat{N}_1-\frac{K_1}{2}\right)\beta_2\hat{Y}_1\frac{r_2}{K_2}\hat{N}_2\delta_0 \\
& +\alpha_1\left\{\beta_1\hat{Y}_2-(1-a)\frac{r_1}{K_1}\hat{Y}_1\right\}\left[\beta_2\hat{Y}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right]\frac{r_2}{K_2}\hat{N}_2\delta_0 \\
& +\alpha_1\beta_1(\hat{N}_1-\hat{Y}_1)\delta_2\, l\left\{\beta_2\hat{Y}_1-(1-a')\frac{r_2}{K_2}\hat{Y}_2\right\}\hat{N}_2 \\
& +\beta_1\hat{Y}_2\frac{2r_1}{K_1}\left(\hat{N}_1-\frac{K_1}{2}\right)\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\frac{r_2}{K_2}\hat{N}_2\delta_0.
\end{aligned}
$$

By using the Routh-Hurwitz criteria, E_5 will be locally asymptotically stable if the following conditions are satisfied.

$$
a_4>0,\quad \begin{vmatrix} a_4 & a_2 \\ 1 & a_3 \end{vmatrix}>0,\quad \begin{vmatrix} a_4 & a_2 & a_0 \\ 1 & a_3 & a_1 \\ 0 & a_4 & a_2 \end{vmatrix}>0,\quad \begin{vmatrix} a_4 & a_2 & a_0 & 0 \\ 1 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_0 \\ 0 & 1 & a_3 & a_1 \end{vmatrix}>0,\quad \begin{vmatrix} a_4 & a_2 & a_0 & 0 & 0 \\ 1 & a_3 & a_1 & 0 & 0 \\ 0 & a_4 & a_2 & a_0 & 0 \\ 0 & 1 & a_3 & a_1 & 0 \\ 0 & 0 & a_4 & a_2 & a_0 \end{vmatrix}>0.
$$

Remark: Here $a_4 > 0$. Also it is noted that if $\hat{N}_1 > \frac{K_1}{2}$, the second inequality is satisfied and $a_0 > 0$.

Nonlinear Analysis by Simulation:

As before, we conjecture that the equilibrium E_5 is globally stable in the interior of the region of attraction. To illustrate this global stability behaviour of E_5 and to see the effect of various parameters on the spread of the disease, the system (4.18) is integrated using the fourth order Runge-Kutta method by taking the same set of parameters as in case I with $Q_0 = Q_a$ and an additional parameter $l = 0.0005$, which satisfy the local stability conditions mentioned above.

The equilibrium value for this set of parameters is

$$\hat{Y}_1 = 1408.005,\ \ \hat{N}_1 = 47531.467,\ \ \hat{Y}_2 = 1817.685,\ \ \hat{N}_2 = 385501.258,\ \ \hat{E} = 43765.735.$$

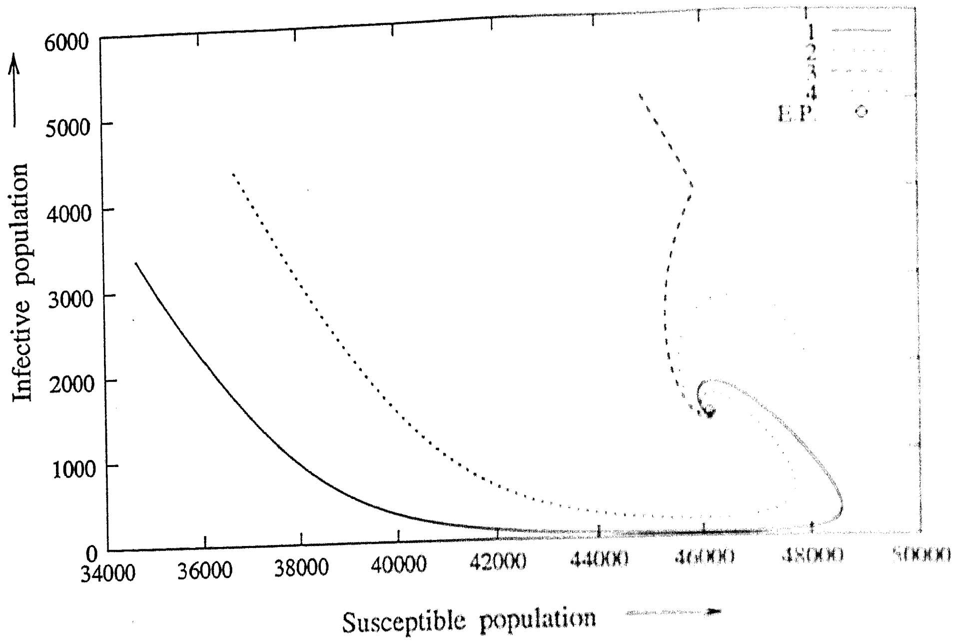


Figure 4.12: Variation of infective population with susceptible population.

In this case also simulation is performed for different initial positions and from the solution curves (Fig. 4.12), we conclude that this equilibrium point is globally stable in the region of attraction T_1, provided that we start away from other four equilibria. Also from Figs. 4.13-4.16, we see that as any of the parameters, r_g, δ_g, Q_0 and I corresponding to the growth of the mosquito population density increases, the infective population increases.

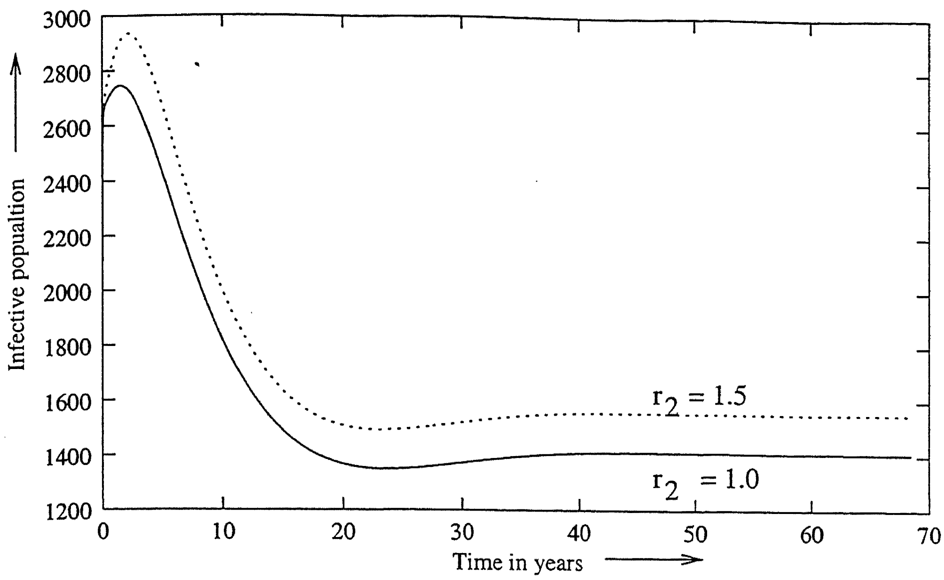


Figure 4.13: Variation of infective population Y_1 with time for different growth rate coefficients of the mosquito population.

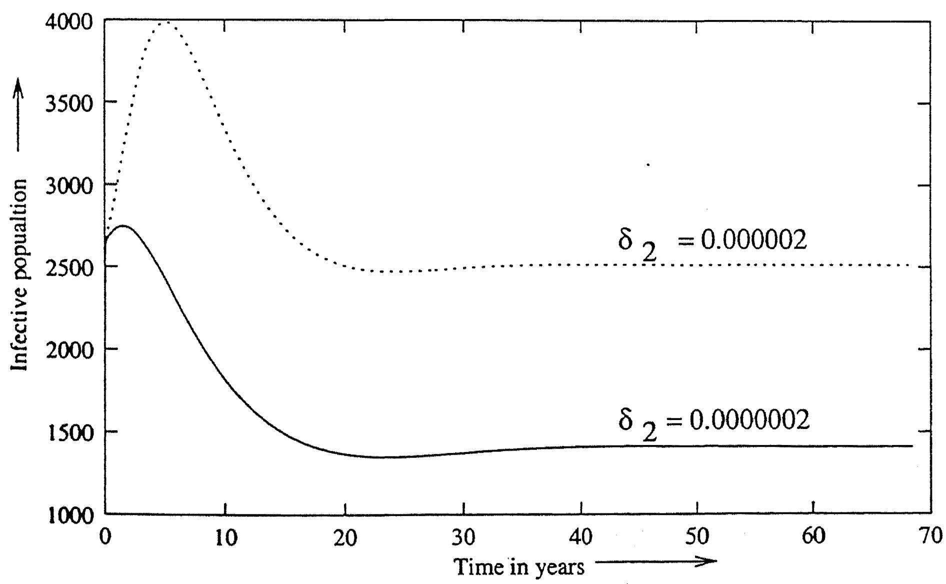


Figure 4.14: Variation of infective population Y_1 with time for different growth rate coefficients of the mosquito population due to environmental discharges.

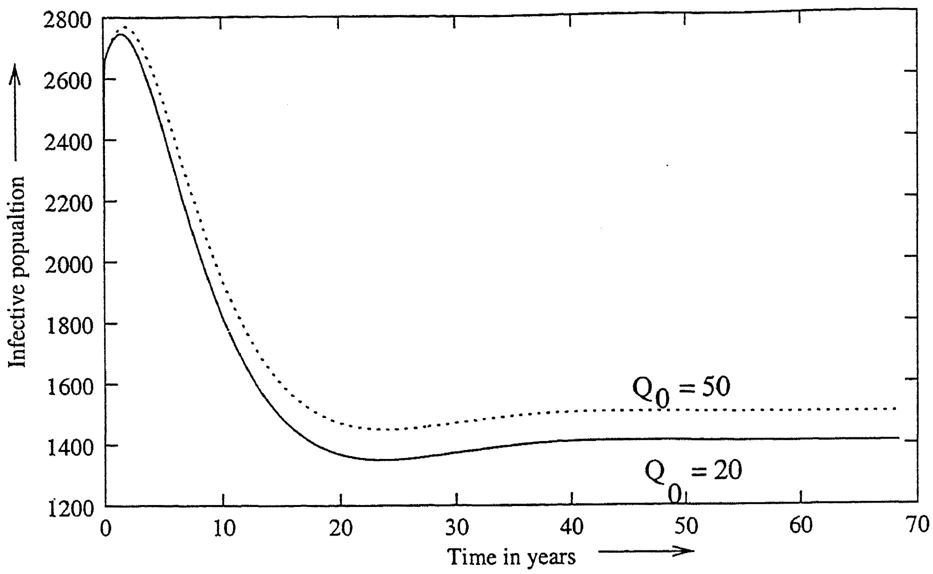


Figure 4.15: Variation of infective population Y_1 with time for different rates of the cumulative discharges.

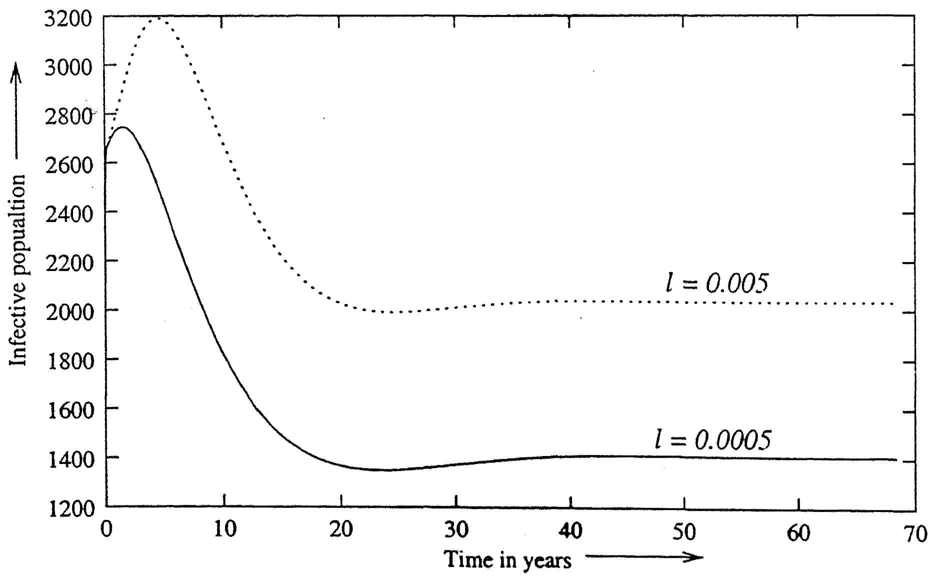


Figure 4.16: Variation of infective population Y_1 with time for different l.

4.4 Conclusions

In this chapter, an SIS model for malaria is proposed and analyzed by considering environmental and demographic effects. The rate of cumulative environmental discharges are assumed to be either a constant or human population dependent. The cases of constant immigration and logistic growth of human population are considered. The growth of the mosquito population is assumed to be logistically varying. Threshold conditions for spread of malaria are derived in each case. If the threshold is greater than one, the nontrivial equilibrium is always feasible. In the case of constant immigration, the nontrivial equilibrium is shown to be locally asymptotically stable when the rate of cumulative environmental discharges is a constant and the same result is also true when the rate of cumulative environmental discharges is human population dependent, but under certain conditions. However, in the case of logistic one, the nontrivial equilibrium is locally asymptotically stable to small perturbations in both the cases of environmental discharges with under certain conditions. By simulation it is also observed that in both types of demographics, these equilibria are in fact globally stable under local stability conditions for the set of parameters considered. Thus, it is concluded that due to household discharges, the mosquito population can grow very large leading to fast spread of malaria. Also, if the rate of immigration increases or the growth rate of human population increases, the infective population increases and the disease becomes more endemic.

Chapter 5

Modelling the Spread of Malaria with Human Reservoir: Environmental and Demographic Effects

5.1 Introduction

The parasites of malaria are normally transmitted from one person to another via female mosquitoes of the Anopheles species. The parasite which infects mosquitoes, one stage of which is called gametocyte, develops in capillaries of the inner organs of infected persons after the invasion of the blood by merozoites. Mature gametocytes, which are infective to mosquitoes, appear in the peripheral blood some three days later in the case of P. vivax and after about ten days later in the case of P. falciparum and P. malariae. The female Anopheles mosquito ingests malaria gametocytes when it takes a blood meal from an infected person. Once a parasite enters into a mosquito, it needs a period of development before it can affect another person again. The length of this period (the sporogonic cycle) depends on the Plasmodium species and the ambient temperature, which is about two weeks.

In highly endemic areas, such as in parts of Africa, persons who have been repeatedly infected by malaria acquire a degree of immunity which suppresses most clinical symptoms. These people carry gametocytes in their blood that infect the mosquitoes biting

them and form a separate class of reservoir population which helps in spreading malaria without getting affected themselves (Nchinda 1998).

Although, as described earlier, usual mathematical models for the spread of malaria by considering different aspects have been proposed and analyzed (Chapter 4), none of the models takes into account the role of the reservoir population as well as environmental effects.

In this chapter, therefore, a mathematical model for the spread of malaria with a reservoir class is proposed and analyzed. The effect of household discharges conducive to the growth of the mosquito population on the spread of malaria is also considered in the model. As in the previous chapters, here also the following two types of population demographics are considered,

(i) a population with constant immigration, and

(ii) a population with logistic growth.

5.2 Malaria Model with Constant Immigration

We consider here an SIS model with immigration, where the human population density $N_1(t)$ is divided into three classes namely, a susceptible class $X_1(t)$, an infective class $Y_1(t)$ and a reservoir class $Z_1(t)$. The mosquito population of density $N_2(t)$ is divided into a susceptible class $X_2(t)$ and an infective class $Y_2(t)$. It is assumed that the rate of growth of density of mosquito follows a generalized logistic model (Gao and Hethcote 1992). In view of the above and by considering the criss-cross interaction of the mosquito population with the human population, a model is proposed as follows,

$$\begin{aligned}
\dot{X}_1 &= A - d_1X_1 - \beta_1X_1Y_2 + \nu_1Y_1, \\
\dot{Y}_1 &= \beta_1X_1Y_2 - (d_1 + \nu_1 + \alpha_1 + \delta_1)Y_1, \\
\dot{Z}_1 &= \delta_1Y_1 - d_1Z_1, \qquad (5.1)\\
\dot{N}_1 &= A - d_1N_1 - \alpha_1Y_1,
\end{aligned}$$

$$\begin{aligned}
\dot{X_2} &= \left(b_2 - a'\frac{r_2N_2}{K_2}\right)N_2 - \left\{d_2 + (1-a')\frac{r_2N_2}{K_2}\right\}X_2 - \beta_2X_2Y_1 \\
&\quad -\lambda_2X_2Z_1 - \alpha_2X_2 + \delta_2N_2E, \\
\dot{Y_2} &= \beta_2X_2Y_1 + \lambda_2X_2Z_1 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2\right\}Y_2, \\
\dot{N_2} &= r_2N_2\left(1 - \frac{N_2}{K_2}\right) - \alpha_2N_2 + \delta_2N_2E, \\
\dot{E} &= Q(N_1) - \delta_0E, \\
N_1 &= X_1 + Y_1 + Z_1, \quad N_2 = X_2 + Y_2, \\
X_1(0) &> 0, \ Y_1(0) \geq 0, \ Z_1(0) \geq 0, \ X_2(0) \geq 0, \ Y_{20} \geq 0, \ E(0) > 0.
\end{aligned}$$

In model (5.1), A is the constant immigration rate of the human population; d_1 is the natural death rate constant; β_1 is the interaction coefficient of susceptible human with the infective mosquito population; ν_1 is the recovery rate coefficient of the human population; α_1 is the disease related death coefficient; δ_1 is the rate coefficient corresponding to movement of human population from the infective class to the reservoir class; $r_2 = b_2 - d_2$ is the growth rate coefficient of the mosquito population; where b_2 and d_2 are birth and death rates constant corresponding to the mosquito population; K_2 is the carrying capacity of the mosquito population in the natural environment; α_2 is the death rate of mosquito population due to control measures, $(\alpha_2 < r_2)$; β_2 and λ_2 are the interaction coefficients of susceptible mosquitoes with the infective and reservoir classes of the human populations respectively; δ_2 is the growth rate coefficient of the mosquito population due to environmental discharges of the cumulative concentration E; $Q(N_1)$ is the cumulative rate of discharges into the environment which depends upon the density of the human population and δ_0 is its natural depletion rate; $0 \leq a' \leq 1$ is a constant, which governs the logistic birth and the logistic death of the mosquito population.

We analyze the model in the following two cases,

(i) the cumulative rate of environmental discharges Q is a constant, and

(ii) Q depends upon density of human population and is considered of the form $Q(N_1) = Q_0 + lN_1$.

5.2.1 Case I : $Q = Q_a$, a Constant

Following the methods as discussed before, the behavior of the system of equation (5.1) can be studied by the following system,

$$\begin{aligned}
\dot{Y_1} &= \beta_1(N_1 - Y_1 - Z_1)Y_2 - (\nu_1 + \alpha_1 + d_1 + \delta_1)Y_1, \\
\dot{Z_1} &= \delta_1 Y_1 - d_1 Z_1, \\
\dot{N_1} &= A - d_1 N_1 - \alpha_1 Y_1, \\
\dot{Y_2} &= \beta_2(\bar{N_2} - Y_2)Y_1 + \lambda_2(\bar{N_2} - Y_2)Z_1 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N_2}\right\} Y_2,
\end{aligned} \tag{5.2}$$

where

$$\bar{N_2} = \limsup_{t\to\infty} N_2 = \frac{K_2}{r_2}\{r_2 - \alpha_2 + \delta_2 \bar{E}\} \text{ and } \bar{E} = \limsup_{t\to\infty} E = \frac{Q_a}{\delta_0}.$$

We see that the region of attraction is

$$T_1 = \left\{(Y_1, Z_1, N_1, Y_2) : 0 \le Y_1 + Z_1 \le N_1 \le \frac{A}{d_1},\ 0 \le Y_2 \le \bar{N_2}\right\}.$$

The result of equilibrium analysis of the system (5.2) is stated in the following theorem. The proof is obvious.

THEOREM 5.1 There exist the following two equilibria corresponding to system (5.2) namely, (i) $P_1(0, 0, \frac{A}{d_1}, 0)$ *and (ii)* $P_2(\hat{Y_1}, \hat{Z_1}, \hat{N_1}, \hat{Y_2})$*. The equilibrium* P_2 *exists if*

$$\frac{\beta_1 \frac{A}{d_1}(\beta_2 + \lambda_2 \frac{\delta_1}{d_1})\bar{N_2}}{\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N_2}\}(\nu_1 + \alpha_1 + d_1 + \delta_1)} = R_s \text{ (say) } > 1 \tag{5.3}$$

where

$$\hat{Y_1} = \frac{\beta_1 A(\beta_2 + \lambda_2 \frac{\delta_1}{d_1})\bar{N_2} - d_1(\nu_1 + \alpha_1 + d_1 + \delta_1)\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N_2}\}}{(\beta_2 + \lambda_2\frac{\delta_1}{d_1})\{\beta_1(\alpha_1 + \delta_1 + d_1)\bar{N_2} + d_1(\nu_1 + \alpha_1 + d_1 + \delta_1)\}}$$

$$\hat{N_1} = \frac{A - \alpha_1 \hat{Y_1}}{d_1}, \quad \hat{Z_1} = \frac{\delta_1 Y_1}{d_1}, \quad \hat{Y_2} = \frac{(\beta_2 + \lambda_2\frac{\delta_1}{d_1})\bar{N_2}\hat{Y_1}}{\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N_2}\} + (\beta_2 + \lambda_2\frac{\delta_1}{d_1})\hat{Y_1}}.$$

Here R_s *is the threshold for the system (5.2).*

5.2.1.1 Stability Analysis

Now we discuss linear stability of both the equilibria and nonlinear stability of only the nontrivial equilibrium. The local stability results of these equilibria are stated in the following theorem.

THEOREM 5.2 The equilibrium P_1 is stable if $R_s > 1$ and unstable if $R_s > 1$ and the equilibrium P_2, if it exists, is locally asymptotically stable.

Proof: The general variational matrix M corresponding to the system (5.2) with respect to (Y_1, Z_1, N_1, Y_2) is given by

$$M = \begin{pmatrix} -(\beta_1 Y_2 + \nu_1 + \alpha_1 + d_1 + \delta_1) & -\beta_1 Y_2 & \beta_1 Y_2 & \beta_1(N_1 - Y_1 - Z_1) \\ \delta_1 & -d_1 & 0 & 0 \\ -\alpha_1 & 0 & -d_1 & 0 \\ \beta_2(\bar{N}_2 - Y_2) & \lambda_2(\bar{N}_2 - Y_2) & 0 & m_{44} \end{pmatrix},$$

where $m_{44} = -[\beta_2 Y_1 + \lambda_2 Z_1 + \{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\}]$.

At the equilibrium point P_1 the variational matrix M_1 is given by

$$M_1 = \begin{pmatrix} -(\nu_1 + \alpha_1 + d_1 + \delta_1) & 0 & 0 & \beta_1 \frac{A}{d_1} \\ \delta_1 & -d_1 & 0 & 0 \\ -\alpha_1 & 0 & -d_1 & 0 \\ \beta_2 \bar{N}_2 & \lambda_2 \bar{N}_2 & 0 & -\left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\} \end{pmatrix}.$$

The characteristic polynomial corresponding to above matrix is given by

$$(d_1 + \psi)\{\psi^3 + a_1\psi^2 + a_2\psi + a_3\} = 0,$$

where

$$\begin{aligned} a_1 &= \nu_1 + \alpha_1 + 2d_1 + \delta_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}, \\ a_2 &= (\nu_1 + \alpha_1 + d_1 + \delta_1)\left[d_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\right] \\ &\quad + d_1\left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\} - \beta_2\bar{N}_2\frac{\beta_1 A}{d_1}, \\ a_3 &= (\nu_1 + \alpha_1 + d_1 + \delta_1)d_1\left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\} - (d_1\beta_2 + \lambda_2\delta_1)\bar{N}_2\frac{\beta_1 A}{d_1}, \end{aligned}$$

and

$$\begin{aligned}
a_1a_2 - a_3 &= \left[\nu_1 + \alpha_1 + 2d_1 + \delta_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\right](\nu_1 + \alpha_1 + d_1 + \delta_1) \\
&\quad \times \left[d_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\right] \\
&\quad + [d_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}]d_1\left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\} \\
&\quad - \beta_1\beta_2\frac{A}{d_1}\bar{N}_2\left[\nu_1 + \alpha_1 + d_1 + \delta_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\right] + \beta_1\frac{A}{d_1}\bar{N}_2\delta_1\lambda_2.
\end{aligned}$$

We note that $a_1a_2 - a_3 > 0$ if $R_s < 1$ and so P_1 is stable. For $R_s > 1$, obviously P_1 is unstable and P_2 exists.

At the equilibrium point P_2, the variational matrix M_2 is given by

$$M_2 = \begin{pmatrix} -(\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + \delta_1) & -\beta_1\hat{Y}_2 & \beta_1\hat{Y}_2 & \beta_1(\hat{N}_1 - \hat{Y}_1 - \hat{Z}_1) \\ \delta_1 & -d_1 & 0 & 0 \\ -\alpha_1 & 0 & -d_1 & 0 \\ \beta_2(\bar{N}_2 - \hat{Y}_2) & \lambda_2(\bar{N}_2 - \hat{Y}_2) & 0 & m_{44} \end{pmatrix},$$

where $m_{44} = -[\beta_2\hat{Y}_1 + \lambda_2 Z_1 + \{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\}]$.

The characteristic polynomial corresponding to above matrix is

$$(d_1 + \psi)\{\psi^3 + g_1\psi^2 + g_2\psi + g_3\} = 0,$$

where

$$\begin{aligned}
g_1 &= d_1 + \beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + \delta_1 + \beta_2\hat{Y}_1 + \lambda_2\hat{Z}_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}, \\
g_2 &= (\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + 2d_1 + \delta_1)\left[\beta_2\hat{Y}_1 + \lambda_2\hat{Z}_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\right] \\
&\quad + d_1(\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + \delta_1) - \beta_1\hat{Y}_2(\delta_1 + \alpha_1) - \beta_1\beta_2(\bar{N}_2 - \hat{Y}_2)(\hat{N}_1 - \hat{Y}_1 - \hat{Z}_1), \\
g_3 &= d_1(\nu_1 + \alpha_1 + d_1 + \delta_1)\left[\beta_2\hat{Y}_1 + \lambda_2\hat{Z}_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\right] \\
&\quad + \beta_1\hat{Y}_2(\delta_1 + \alpha_1)\left[\beta_2\hat{Y}_1 + \lambda_2\hat{Z}_1 + \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\right] \\
&\quad - \beta_1(\hat{N}_1 - \hat{Y}_1 - \hat{Z}_1)(\bar{N}_2 - \hat{Y}_2)(\delta_1\lambda_2 + d_1\beta_2).
\end{aligned}$$

It can be checked that $g_1g_2 - g_3 > 0$. Hence by the Routh-Hurwitz criteria the equilibrium point P_2 is locally asymptotically stable.

Nonlinear Analysis and Simulation: It is noted that system (5.2) is bounded by the system (5.2) with $\alpha_1 = 0,\ \alpha_2 = 0$, which on using comparison theorems (Lakshmikantham

and Leela 1969) implies that the solution of (5.2) is bounded by the solution of (5.2) with $\alpha_i = 0$, for $i = 1, 2$. Using a heuristic method by setting $Z_1 = \frac{d_1}{\delta_1} Y_1$ the system (5.2) with $\alpha_1 = 0$, $\alpha_2 = 0$ can be shown to be globally stable (see Appendix I). Hence we speculate that the nontrivial equilibrium point of the system (5.2) may be globally stable if we start away from P_1. To further illustrate this, and to see the effects of various parameters on the equilibrium level of the infective population, the system (5.2) is integrated using the fourth order Runge-Kutta Method by using following set of parameters (Greenhalgh 1990);

$$\beta_1 = 0.00000022 = \beta_2,\ \nu_1 = 0.012,\ \alpha_1 = 0.0005,\ d_1 = 0.0004,$$

$$\delta_1 = 0.00002,\ A = 10,\ \lambda_2 = 0.00000011,\ \alpha_2 = 0.045,\ r_2 = 1,$$

$$d_2 = 0.04,\ \ a' = 0.999,\ \delta_2 = 0.0000002,\ Q_0 = 20,\ \delta_0 = 0.001,\ K_2 = 1000000.$$

The equilibrium values of $\hat{Y}_1$, $\hat{Z}_1$, $\hat{N}_1$ and $\hat{Y}_2$ corresponding to P_2 are as follows,

$$\hat{Y}_1 = 701.53, \quad \hat{Z}_1 = 35.077, \quad \hat{N}_1 = 24123.07, \quad \hat{Y}_2 = 1761.68.$$

Simulation of (5.2) is performed for different initial positions 1, 2, 3 and 4 as shown in Fig. 5.1. In Fig. 5.1, the infective population is plotted against the susceptible population, which indicates that it is plausible for the system to be globally stable about this equilibrium point P_2 provided we start away from other equilibrium point. Further to see the effects of environmental discharges and immigration on the equilibrium level of the infective population, in Figs. 5.2 and 5.3, the infective population is plotted against time for different Q_a, δ_1 and A. From these figures, we conclude that the endemic equilibrium level of the infective population increases as any of A, $\bar{N}_2$ and δ_1, increases.

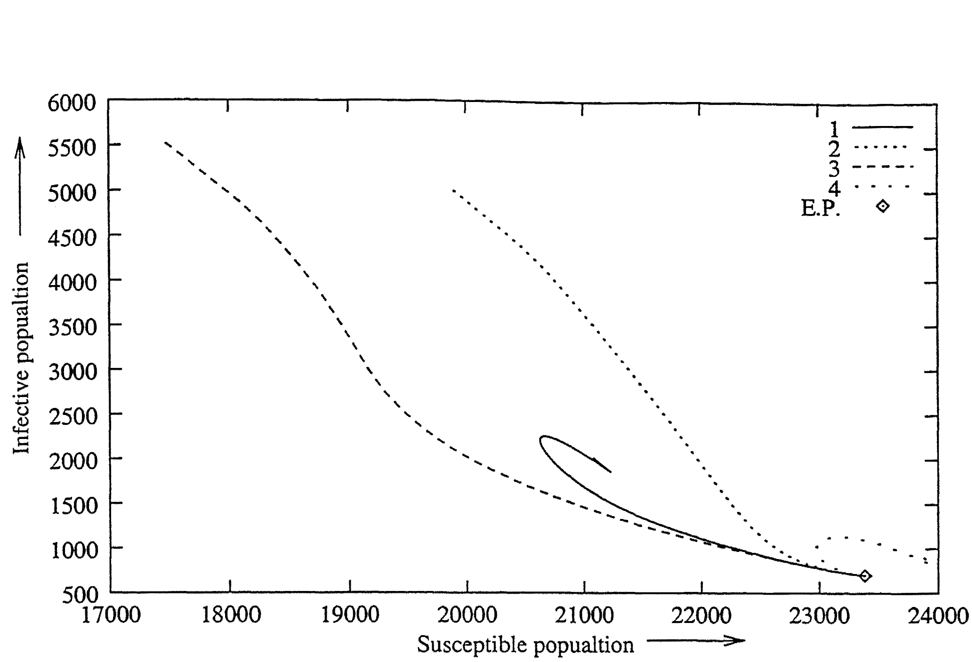


Figure 5.1: Variation of infective human population with susceptible human population.

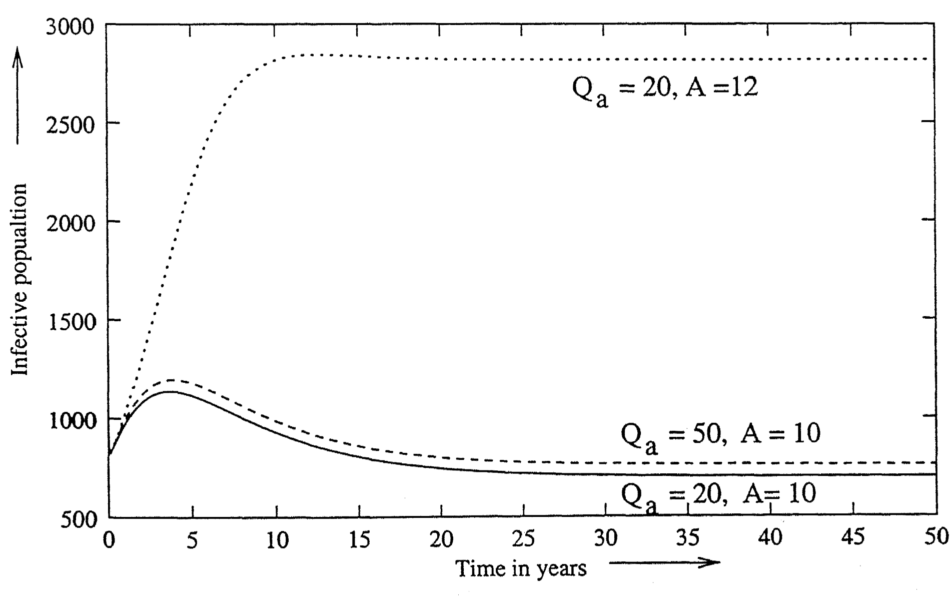


Figure 5.2: Variation of infective human population with time for different rates of immigration and different rates of cumulative environmental discharges.

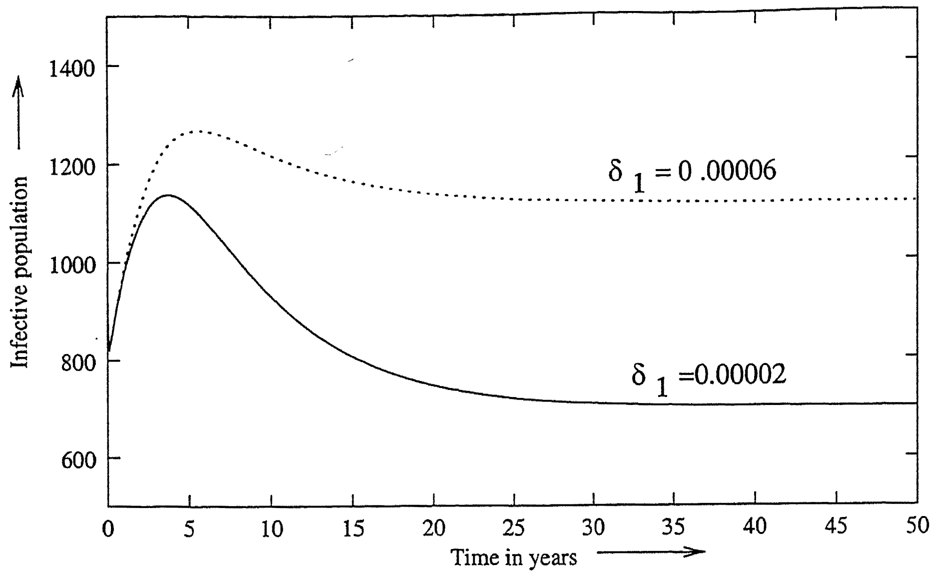


Figure 5.3: Variation of infective human population with time for different rate coefficients corresponding to movement of human population from infective class to reservoir class.

5.2.2 Case II: $Q = Q_0 + lN_1$

In this case it is sufficient to consider the following subsystem of the system (5.1).

$$\begin{aligned}
\dot{Y_1} &= \beta_1(N_1 - Y_1 - Z_1)Y_2 - (\nu_1 + \alpha_1 + d_1 + \delta_1)Y_1, \\
\dot{Z_1} &= \delta_1 Y_1 - d_1 Z_1, \\
\dot{N_1} &= A - d_1 N_1 - \alpha_1 Y_1, \\
\dot{Y_2} &= \beta_2(N_2 - Y_2)Y_1 + \lambda_2(N_2 - Y_2)Z_1 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2\right\}Y_2, \\
\dot{N_2} &= r_2 N_2\left(1 - \frac{N_2}{K_2}\right) - \alpha_2 N_2 + \delta_2 N_2 E, \\
\dot{E} &= Q_0 + lN_1 - \delta_0 E.
\end{aligned} \tag{5.4}$$

In this case the region of attraction is

$$\begin{aligned}
T_1' = & \left\{(Y_1, Z_1, N_1, Y_2, N_2, E) : 0 \le Y_1 + Z_1 \le N_1 \le \frac{A}{d_1},\right. \\
& \left. 0 \le Y_2 \le N_2 \le \frac{K_2}{r_2}\left(r_2 - \alpha_2 + \delta_2\frac{Q(\frac{A}{d_1})}{\delta_0}\right),\ 0 \le E \le \frac{Q(\frac{A}{d_1})}{\delta_0}\right\},
\end{aligned}$$

and the model is well posed. The result of equilibrium analysis of the system (5.4) is stated in the following theorem.

THEOREM 5.3 There exist following three equilibria corresponding to system (5.4) namely, (i)$\bar{P}_1\left(0,0,\frac{A}{d_1},0,0,\frac{(Q_0+lN_1)}{\delta_0}\right)$, (ii) $\bar{P}_2\left(0,0,\frac{A}{d_1},0,N_2{}^*,E^*\right)$,
where,

$$N_2^* = \frac{K_2}{r_2}\{r_2-\alpha_2+\delta_2E^*\} \ > \ 0, \quad E^* = \frac{Q_0+l\frac{A}{d_1}}{\delta_0},$$

and (iii) $\bar{P}_3(\hat{Y}_1,\hat{Z}_1,\hat{N}_1,\hat{Y}_2,\hat{N}_2,\hat{E})$, *which exists if*

$$\frac{\beta_1\frac{A}{d_1}(\beta_2+\lambda_2\frac{\delta_1}{d_1})N_2^*}{(\nu_1+\alpha_1+d_1+\delta_1)\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}N_2^*\}} = R_s'(say) \ > \ 1,$$

here R_s' *is the threshold of the system (5.4).*

Proof: The existence of either of the first two equilibrium points is obvious, we give here the proof of existence of only the third equilibrium point.

Equating right hand side of (5.4) to zero, we get the following after some calculations,

$$Z_1 = \frac{\delta_1}{d_1}Y_1, \quad N_1 = \frac{A-\alpha_1Y_1}{d_1}, \quad N_2 = \frac{K_2}{r_2}\{r_2-\alpha_2+\delta_2E\}, \quad E = \frac{Q_0+lN_1}{\delta_0}, \tag{5.5}$$

and

$$Y_2 = \frac{d_1(\nu_1+\alpha_1+d_1+\delta_1)Y_1}{\beta_1[A-(d_1+\alpha_1+\delta_1)Y_1]}. \tag{5.6}$$

Also we have

$$Y_2 = \frac{\left[\frac{K_2}{r_2}\{r_2-\alpha_2+\frac{\delta_2}{\delta_0}(Q_0+l\frac{A}{d_1})-\frac{\delta_2}{\delta_0}\frac{l\alpha_1}{d_1}Y_1\}\right](\beta_2+\lambda_2\frac{\delta_1}{d_1})Y_1}{\left[\alpha_2+d_2+(1-a')\{r_2-\alpha_2+\frac{\delta_2}{\delta_0}(Q_0+l\frac{A}{d_1})-\frac{\delta_2}{\delta_0}\frac{l\alpha_1}{d_1}Y_1\}\right]+(\beta_2+\lambda_2\frac{\delta_1}{d_1})Y_1}. \tag{5.7}$$

For $Y_1\neq 0$, (5.6) and (5.7) gives following quadratic in Y_1,

$$H_1Y_1^2+H_2Y_1+H_3=0, \tag{5.8}$$

where

$$\begin{aligned}
H_1 &= \beta_1(\beta_2+\lambda_2\frac{\delta_1}{d_1})\left(1+\frac{\alpha_1+\delta_1}{d_1}\right)\frac{K_2\delta_2l\alpha_1}{r_2\delta_0d_1},\\
H_2 &= -\Bigg[\beta_1(\beta_2+\lambda_2\frac{\delta_1}{d_1})(1+\frac{\alpha_1+\delta_1}{d_1})N_2^*+\beta_1\frac{A}{d_1}\frac{K_2\delta_2l\alpha_1}{r_2\delta_0d_1}(\beta_2+\lambda_2\frac{\delta_1}{d_1})\\
&\quad +(\nu_1+\alpha_1+d_1+\delta_1)\left\{(\beta_2+\lambda_2\frac{\delta_1}{d_1})-(1-a')\frac{\delta_2l\alpha_1}{\delta_0d_1}\right\}\Bigg],\\
H_3 &= \beta_1(\beta_2+\lambda_2\frac{\delta_1}{d_1})\frac{A}{d_1}N_2^*-(\nu_1+\alpha_1+d_1+\delta_1)\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}N_2^*\right\}.
\end{aligned}$$

From (5.8), we note the following,

(i) $F(0) = \beta_1(\beta_2 + \lambda_2 \frac{\delta_1}{d_1})\frac{A}{d_1}N_2^* - \{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\}(\nu_1 + \alpha_1 + d_1 + \delta_1)$,

(ii) $F\left(\frac{A}{\alpha_1+d_1+\delta_1}\right) < 0$, and

(iii) $F(\infty) > 0$.

So if $F(0) > 0$, then there exists one root say $\hat{Y}_1$ between 0 and $\frac{A}{\alpha_1+d_1+\delta_1}$ and another root between $\frac{A}{\alpha_1+d_1+\delta_1}$ and ∞. Corresponding to this $\hat{Y}_1$, the values of $\hat{N}_1$, $\hat{Y}_2$, $\hat{N}_2$ and $\hat{E}$ are determined using (5.5) and (5.6) and we get a nontrivial equilibrium point $\bar{E}_3$ under following condition

$$\frac{\beta_1 \frac{A}{d_1}(\beta_2 + \lambda_2 \frac{\delta_1}{d_1})N_2^*}{(\nu_1 + \alpha_1 + d_1 + \delta_1)\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\}} = R_s'(\text{say}) \ > \ 1.$$

As in previous cases here also R_s' is the threshold of the system (5.4). Using (5.5) and (5.6), values of $\hat{Y}_2$, $\hat{Z}_1$, $\hat{N}_1$, $\hat{N}_2$, $\hat{E}$ can be derived corresponding to $\hat{Y}_1$.

5.2.2.1 Stability Analysis

We state the stability result of these equilibria P_1, P_2 and P_3 in the following theorem.

THEOREM 5.4 The equilibrium $\bar{P}_1$ is unstable, the equilibrium $\bar{P}_2$ is stable if $R_s' < 1$, otherwise if $R_s' > 1$ it is unstable and the equilibrium $\bar{P}_3$ exists, which is stable under the following conditions:

$$\begin{vmatrix} a_4 & a_2 & a_0 \\ 1 & a_3 & a_1 \\ 0 & a_4 & a_2 \end{vmatrix} > 0 \ \textit{and} \ \begin{vmatrix} a_4 & a_2 & a_0 & 0 \\ 1 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_0 \\ 0 & 1 & a_3 & a_1 \end{vmatrix} > 0,$$

where a_4, a_3, a_2, a_1 and a_0 are given explicitly in the proof of the theorem.

Proof: The variational matrix at $(Y_1, Z_1, N_1, Y_2, N_2, E)$ is given by

$$M = \begin{pmatrix} m_{11} & -\beta_1 Y_2 & \beta_1 Y_2 & \beta_1(N_1 - Y_1 - Z_1) & 0 & 0 \\ \delta_1 & -d_1 & 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & -d_1 & 0 & 0 & 0 \\ \beta_2(N_2 - Y_2) & \lambda_2(N_2 - Y_2) & 0 & m_{44} & m_{45} & 0 \\ 0 & 0 & 0 & 0 & m_{55} & \delta_2 N_2 \\ 0 & 0 & l & 0 & 0 & -\delta_0 \end{pmatrix},$$

where $m_{11} = -(\beta_1 Y_2 + \nu_1 + \alpha_1 + d_1 + \delta_1)$, $m_{44} = -[\beta_2 Y_1 + \lambda_2 Z_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2} N_2]$, $m_{45} = \beta_2 Y_1 + \lambda_2 Z_1 - (1-a')\frac{r_2}{K_2} Y_2$ and $m_{55} = r_2 - \alpha_2 + \delta_2 E - \frac{2r_2}{K_2} N_2$.

The variational matrix M_1 at $\bar{P}_1$ is given by

$$M_1 = \begin{pmatrix} -(\nu_1+\alpha_1+d_1+\delta_1) & 0 & 0 & \frac{\beta_1 A}{d_1} & 0 & 0 \\ \delta_1 & -d_1 & 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & -d_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(\alpha_2+d_2) & 0 & 0 \\ 0 & 0 & 0 & 0 & r_2-\alpha_2+\delta_2 E^* & 0 \\ 0 & 0 & l & 0 & 0 & -\delta_0 \end{pmatrix},$$

Clearly one root is positive (as $r_2 > \alpha_2$) implying instability of this equilibrium point.

The variational matrix M_2 at equilibrium point $\bar{P}_2$ is

$$M_2 = \begin{pmatrix} -(\nu_1+\alpha_1+d_1+\delta_1) & 0 & 0 & \frac{\beta_1 A}{d_1} & 0 & 0 \\ \delta_1 & -d_1 & 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & -d_1 & 0 & 0 & 0 \\ \beta_2 N_2^* & \lambda_2 N_2^* & 0 & \bar{m}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & -(r_2-\alpha_2+\delta_2 E^*) & \delta_2 N_2^* \\ 0 & 0 & l & 0 & 0 & -\delta_0 \end{pmatrix},$$

where $\bar{m}_{44} = -\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2} N_2^*\}$.

The characteristic equation corresponding to matrix M_2 is given by

$$(d_1+\psi)(r_2-\alpha_2+\delta_2 E^*+\psi)(\delta_0+\psi)\{\psi^3+g_1\psi^2+g_2\psi+g_3\} = 0,$$

where

$$\begin{aligned} g_1 &= \nu_1+\alpha_1+2d_1+\delta_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}N_2^*\right\} > 0, \\ g_2 &= (\nu_1+\alpha_1+d_1+\delta_1)\left[d_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}N_2^*\right\}\right] \\ &\quad +d_1\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}N_2^*\right\} - \beta_1\frac{A}{d_1}\beta_2 N_2^*, \\ g_3 &= \left[d_1\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}N_2^*\right\}(\nu_1+\alpha_1+d_1+\delta_1) - \beta_1 A N_2^*(\beta_2+\lambda_2\frac{\delta_1}{d_1})\right]. \end{aligned}$$

Clearly three roots of the above polynomial are negative, while other roots are given by the cubic equation. For no positive root to exist, we must have g_3 positive, which gives $R_s' < 1$. Further,

$$\begin{aligned} g_1 g_2 - g_3 &= \left[\nu_1+\alpha_1+2d_1+\delta_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}N_2^*\right\}\right] \\ &\quad \times\left(d_1\left[\nu_1+\alpha_1+d_1+\delta_1+\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}N_2^*\}\right]\right. \\ &+ \left.\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}N_2^*\right\}(\nu_1+\alpha_1+d_1+\delta_1) - \beta_1\beta_2\frac{A}{d_1}N_2^*\right) \end{aligned}$$

$$-\left[d_1\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}N_2^*\right\}(\nu_1+\alpha_1+d_1+\delta_1)-\beta_1AN_2^*(\beta_2+\lambda_2\frac{\delta_1}{d_1})\right]>0.$$

Hence by the Routh-Hurwitz criteria this equilibrium point is stable under the condition $R_s'<1$ and if $R_s'>1$ it is unstable and the third equilibrium point exists.

The variational matrix M_3 at equilibrium point $\bar{P}_3$ is given by

$$M_3=\begin{pmatrix} \hat{m}_{11} & -\beta_1\hat{Y}_2 & \beta_1\hat{Y}_2 & \beta_1(\hat{N}_1-\hat{Y}_1-\hat{Z}_1) & 0 & 0\\ \delta_1 & -d_1 & 0 & 0 & 0 & 0\\ -\alpha_1 & 0 & -d_1 & 0 & 0 & 0\\ \beta_2(\hat{N}_2-\hat{Y}_2) & \lambda_2(\hat{N}_2-\hat{Y}_2) & 0 & \hat{m}_{44} & \hat{m}_{45} & 0\\ 0 & 0 & 0 & 0 & \hat{m}_{55} & \delta_2\hat{N}_2\\ 0 & 0 & l & 0 & 0 & -\delta_0 \end{pmatrix},$$

where

$\hat{m}_{11}=-(\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+\delta_1)$, $\hat{m}_{44}=-\left[\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\right]$,
$\hat{m}_{45}=(\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1)-(1-a')\frac{r_2}{K_2}\hat{Y}_2$ and $\hat{m}_{55}=-\{r_2-\alpha_2+\delta_2\hat{E}\}$.

From the characteristic polynomial corresponding to matrix M_3, we see that one root of the above matrix is $-d_1$ and other roots are given by the following polynomial,

$$\psi^5+a_4\psi^4+a_3\psi^3+a_2\psi^2+a_1\psi+a_0=0,$$

where

$$\begin{aligned}
a_4 &= \delta_0+r_2-\alpha_2+\delta_2\hat{E}+\beta_1\hat{Y}_2+\nu_1+\alpha_1+2d_1+\delta_1+\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1\\
&\quad+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\},\\
a_3 &= \delta_0\Big[r_2-\alpha_2+\delta_2\hat{E}+\beta_1\hat{Y}_2+\nu_1+\alpha_1+2d_1+\delta_1+\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1\\
&\quad+\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\}\Big]+(r_2-\alpha_2+\delta_2\hat{E})\Big[\beta_1\hat{Y}_2+\nu_1+\alpha_1+2d_1\\
&\quad+\delta_1+\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\Big]\\
&\quad+(\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+\delta_1)\left[\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}+d_1\right]\\
&\quad+(\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\alpha_2)d_1+\beta_1\hat{Y}_2(\delta_1+\alpha_1)+\beta_1\beta_2(\hat{N}_1-\hat{Y}_1-\hat{Z}_1)(\hat{N}_2-\hat{Y}_2),\\
a_2 &= \delta_0(r_2-\alpha_2+\delta_2\hat{E})\Big[\beta_1\hat{Y}_2+\nu_1+\alpha_1+2d_1+\delta_1+\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\\
&\quad\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}+d_1\Big]+(r_2-\alpha_2+\delta_2\hat{E})(\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+\delta_1)\\
&\quad\times\left[\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\}+d_1\right]\\
&\quad+(\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+\delta_1)\left[\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}+d_1\right](d_1+\delta_0)\\
&\quad+\left[\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\right]d_1(\delta_0+r_2-\alpha_2+\delta_2\hat{E})
\end{aligned}$$

$$
\begin{aligned}
& +d_1\delta_0(\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+\delta_1) \\
& +\beta_1\hat{Y}_2(\delta_1+\alpha_1)\left[\delta_0+r_2-\alpha_2+\delta_2\hat{E}+\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\right] \\
& +\beta_1(\hat{N}_1-\hat{Y}_1-\hat{Z}_1)(\hat{N}_2-\hat{Y}_2)\left\{\delta_1\lambda_2+\beta_2(\delta_0+d_1+r_2-\alpha_2+\delta_2\hat{E})\right\}, \\
a_1 \;=\; & \delta_0(r_2-\alpha_2+\delta_2\hat{E})(\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+\delta_1)\left[\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\right] \\
& +(r_2-\alpha_2+\delta_2\hat{E})(\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+\delta_1)\left[\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\right]d_1 \\
& +(\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+\delta_1)\left[\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\right]d_1\delta_0 \\
& +\left[\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\right]d_1\delta_0(r_2-\alpha_2+\delta_2\hat{E}) \\
& +d_1\delta_0(r_2-\alpha_2+\delta_2\hat{E})(\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+\delta_1) \\
& +\beta_1\hat{Y}_2(\delta_1+\alpha_1)\left[\left\{\delta_0+\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}(r_2-\alpha_2+\delta_2\hat{E})\right. \\
& \left.+\delta_0\{\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\}\right] \\
& +\beta_1(\hat{N}_1-\hat{Y}_1-\hat{Z}_1)(\hat{N}_2-\hat{Y}_2)\left[\delta_1\lambda_2(\delta_0+r_2-\alpha_2+\delta_2\hat{E})\right. \\
& \left.+\beta_2\left\{\delta_0 d_1+(d_1+\delta_0)(r_2-\alpha_2+\delta_2\hat{E})\right\}\right], \\
a_0 \;=\; & \delta_0(r_2-\alpha_2+\delta_2\hat{E})(\beta_1\hat{Y}_2+\nu_1+\alpha_1+d_1+\delta_1)\left[\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\right]d_1 \\
& +\beta_1\hat{Y}_2(\delta_1+\alpha_1)\delta_0(r_2-\alpha_2+\delta_2\hat{E})\left[\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\hat{N}_2\right\}\right] \\
& +\beta_1(\hat{N}_1-\hat{Y}_1-\hat{Z}_1)(\hat{N}_2-\hat{Y}_2)(r_2-\alpha_2+\delta_2\hat{E})\delta_0(\delta_1\lambda_2+\beta_2 d_1) \\
& +\delta_2\hat{N}_2\left[\beta_2\hat{Y}_1+\lambda_2\hat{Z}_1-(1-a')\frac{r_2}{K_2}\hat{Y}_2\right]\beta_1(\hat{N}_1-\hat{Y}_1-\hat{Z}_1)\alpha_1 l.
\end{aligned}
$$

Using the Routh-Hurwitz criteria, we see that this equilibrium point $\bar{P}_3$ is locally asymptotically stable if the following conditions are satisfied,

$$
a_4>0,\quad \begin{vmatrix} a_4 & a_2 \\ 1 & a_3 \end{vmatrix}>0,\quad \begin{vmatrix} a_4 & a_2 & a_0 \\ 1 & a_3 & a_1 \\ 0 & a_4 & a_2 \end{vmatrix}>0,\quad \begin{vmatrix} a_4 & a_2 & a_0 & 0 \\ 1 & a_3 & a_1 & 0 \\ 0 & a_4 & a_2 & a_0 \\ 0 & 1 & a_3 & a_1 \end{vmatrix}>0,\quad \begin{vmatrix} a_4 & a_2 & a_0 & 0 & 0 \\ 1 & a_3 & a_1 & 0 & 0 \\ 0 & a_4 & a_2 & a_0 & 0 \\ 0 & 1 & a_3 & a_1 & 0 \\ 0 & 0 & a_4 & a_2 & a_0 \end{vmatrix}>0.
$$

The first two conditions are obviously true. If the next two are satisfied then so is the fifth. Hence the equilibrium $\bar{P}_3$ is locally asymptotically stable under the conditions mentioned in the theorem.

Nonlinear Analysis and Simulation: We note that the system (5.4) is bounded by its corresponding system when $\alpha_1 = 0$ in T_1'. As in the earlier case using heuristic approach the system (5.4) with $\alpha_1 = 0$ can be shown to be globally stable about the non-trivial

equilibrium point provided we do not start from the other equilibria (see Appendix II). Hence we speculate that $\bar{P}_3$ may be globally stable provided that we start away from other equilibria.

To show this, the system (5.4) is integrated using the fourth order Runge-Kutta method using previous values of parameters with $Q_0 = Q_a$ and an additional parameter $l =$ 0.0005, which satisfy the local stability conditions mentioned in Theorem 5.4.

The equilibrium values of $\hat{Y}_1$, $\hat{Z}_1$, $\hat{N}_1$, $\hat{Y}_2, \hat{N}_2$ and $\hat{E}$ are as follows,

$$\hat{Y}_1 = 4181.8, \quad \hat{Z}_1 = 209.2, \quad \hat{N}_1 = 19771.8, \; \hat{Y}_2 = 14051.1, \quad \hat{N}_2 = 960977.1, \quad \hat{E} = 29885.6.$$

Simulation of the system (5.4) has been performed for different initial positions 1, 2, 3 and 4 as shown in Fig. 5.4. In this figure, the infective population is plotted against the susceptible population and from the solution curves, we conclude that the system is globally stable about this equilibrium point $\bar{P}_3$ under the set of parameters considered. Also from Figs. 5.5-5.9, where the infective population is plotted with time for different r_2, δ_2, Q_0, l, and δ_1, we note that the infective population increases as any of these parameters increases, which correspond to the growth of the mosquito population and environmental discharges.

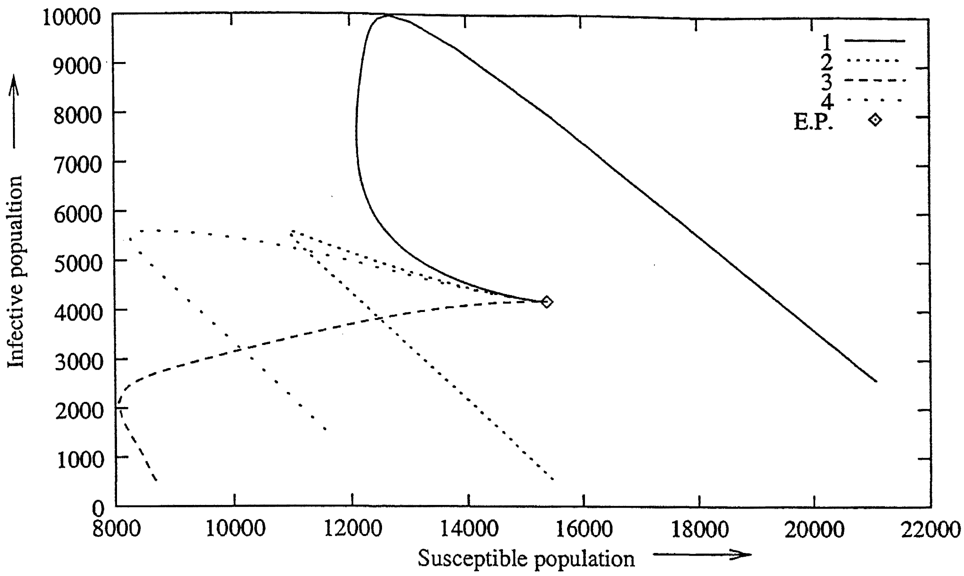


Figure 5.4: Variation of infective human population with susceptible human population.

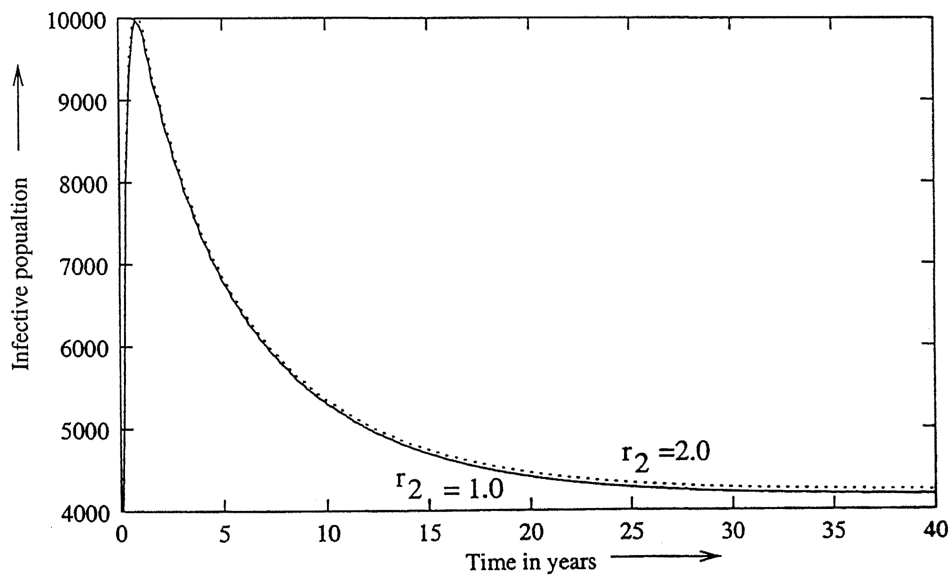


Figure 5.5: Variation of infective human population with time for different intrinsic growth rates of mosquito population.

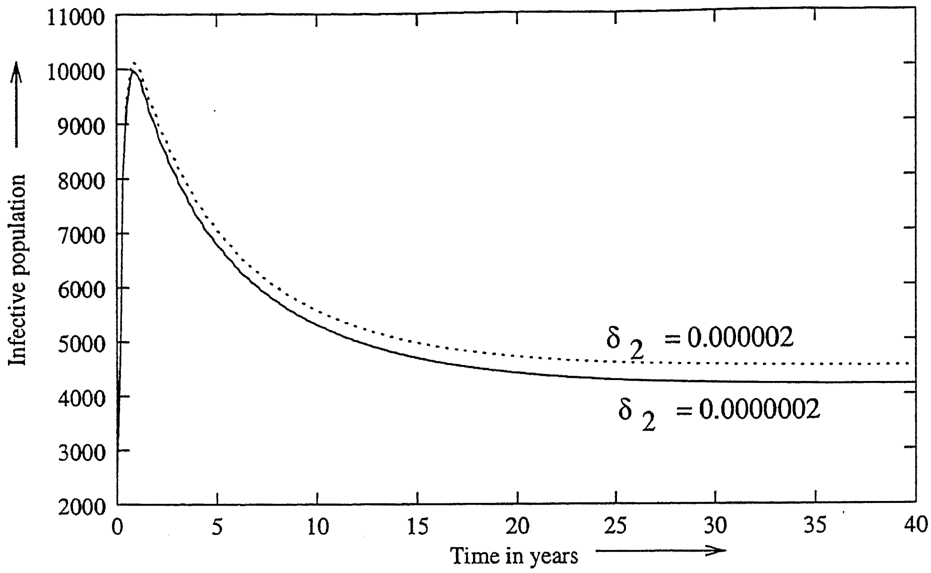


Figure 5.6: Variation of infective human population with time for different growth rate coefficients of mosquito population due to environmental discharges.

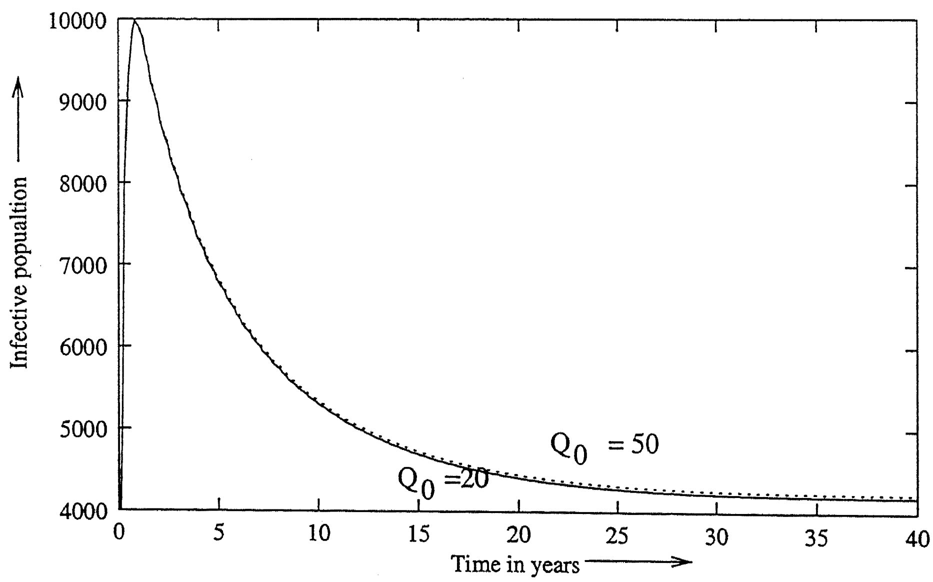


Figure 5.7: Variation of infective human population with time for different rates of cumulative environmental discharges.

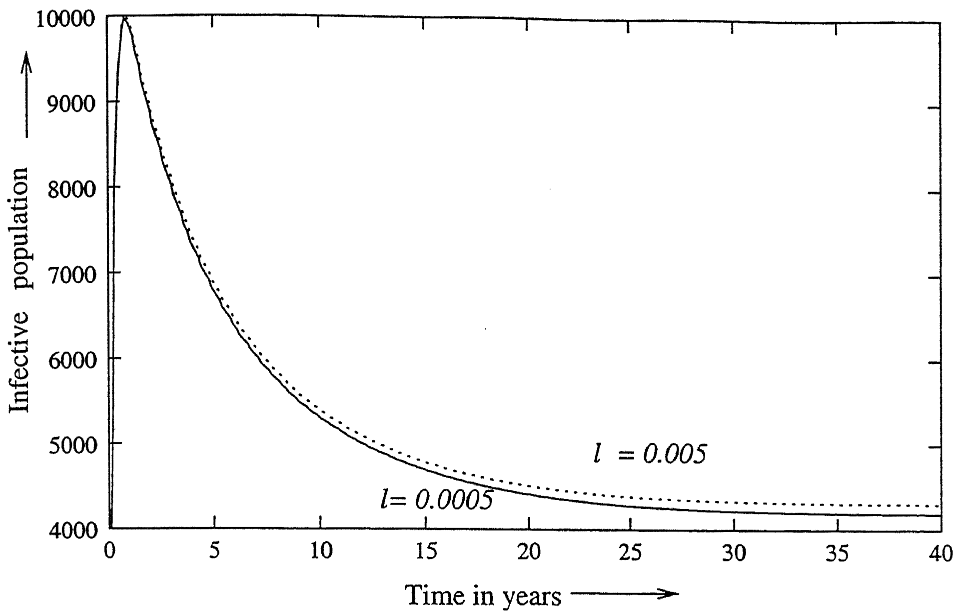


Figure 5.8: Variation of infective human population with time for different l.

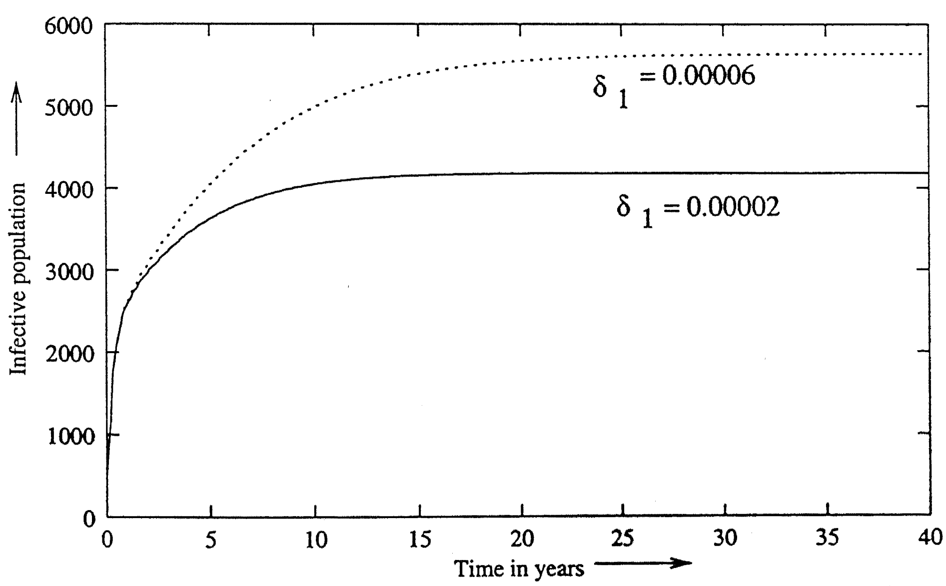


Figure 5.9: Variation of infective human population with time for different rate coefficients corresponding to movement of the human population from the infective class to the reservoir class.

5.3 Malaria Model with Logistic Population Growth

Now let us consider an SIS model where the human population growth is logistic so that both the birth and death rates are density dependent in such a manner that the birth rate decreases and death rate increases as the population size increases towards its carrying capacity. Keeping in view the other considerations discussed in previous section, a mathematical model is proposed as follows:

$$
\begin{aligned}
\dot{X}_1 &= \left[b_1 - a\frac{r_1}{K_1}N_1\right]N_1 - \left[d_1 + (1-a)\frac{r_1}{K_1}N_1\right]X_1 - \beta_1 X_1 Y_2 + \nu_1 Y_1, \\
\dot{Y}_1 &= \beta_1 X_1 Y_2 - \left[\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)\frac{r_1 N_1}{K_1}\right]Y_1, \\
\dot{Z}_1 &= \delta_1 Y_1 - \{d_1 + (1-a)\frac{r_1}{K_1}N_1\}Z_1, \\
\dot{N}_1 &= r_1(1 - \frac{N_1}{K_1}) - \alpha_1 Y_1, \qquad (5.9)\\
\dot{X}_2 &= (b_2 - a'\frac{r_2}{K_2}N_2)N_2 - \{d_2 + (1-a')\frac{r_2}{K_2}N_2\}X_2 - \beta_2 X_2 Y_1 - \lambda_2 X_2 Z_1 - \alpha_2 Y_2, \\
\dot{Y}_2 &= \beta_2 X_2 Y_1 + \lambda_2 X_2 Z_1 - \{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2\}Y_2, \\
\dot{N}_2 &= r_2 N_2(1 - \frac{N_2}{K_2}) - \alpha_2 N_2 + \delta_2 N_2 E, \\
\dot{E} &= Q(N_1) - \delta_0 E, \\
N_1 &= X_1 + Y_1 + Z_1, \quad N_2 = X_2 + Y_2,\ r_2 > \alpha_2,\ \delta_1 < d_1,\ 0 \le a \le 1, \quad 0 \le a' \le 1,
\end{aligned}
$$

$$X_1(0) = X_{10} > 0,\ Y_1(0) = Y_{10} \ge 0,\ Z_1(0) = Z_{10} \ge 0\ , X_2(0) = X_{20} \ge 0,\ Y_{20} \ge 0,\ E(0) = E_0 > 0.$$

Here b_1 and d_1 are the natural birth and death rates, $r_1 = b_1 - d_1$ is the growth rate constant; r_1 and K_1 are the logistic growth rate and carrying capacity of the environment corresponding to human population; r_2 and K_2 are the logistic growth rate and carrying capacity of the mosquito population in the environment. All other parameters are defined in the previous section. For $0 < a < 1$, the birth rate decreases and the death rate increases as N_1 increases to its carrying capacity K_1. When $a = 1$, the model could be called simply a logistic birth model as all of the restricted growth is due to a decreasing birth rate and the death rate is constant. Similarly, when $a = 0$, it could be called a logistic death model as all of the restricted growth is due to an increasing death rate and

the birth rate is constant.

As in the previous section, here also the following two cases are considered,

(i) the rate of cumulative environmental discharges Q is a constant, and

(ii) Q is human population density dependent.

5.3.1 Case I: $Q = Q_a$, a Constant

In this case since $X_1+Y_1+Z_1 = N_1$ and $X_2+Y_2 = Z_2$, the behaviour of the system (5.9) can be studied by the following system,

$$\begin{aligned}
\dot{Y_1} &= \beta_1[N_1 - Y_1 - Z_1]Y_2 - \left[\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)\frac{r_1 N_1}{K_1}\right] Y_1, \\
\dot{Z_1} &= \delta_1 Y_1 - \left\{d_1 + (1-a)\frac{r_1}{K_1}N_1\right\} Z_1, \\
\dot{N_1} &= r_1 N_1 (1 - \frac{N_1}{K_1}) - \alpha_1 Y_1, \\
\dot{Y_2} &= \beta_2(\bar{N}_2 - Y_2)Y_1 + \lambda_2(\bar{N}_2 - Y_2)Z_1 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\right\} Y_2,
\end{aligned} \tag{5.10}$$

where as in the previous section,

$$\bar{N}_2 = \limsup_{t\to\infty} N_2 = \frac{K_2}{r_2}\{r_2 - \alpha_2 + \delta_0 \bar{E}\} \text{ and } \bar{E} = \limsup_{t\to\infty} E = \frac{Q_a}{\delta_0}.$$

The results of an equilibrium analysis of the system (5.10) are stated in the following theorem.

THEOREM 5.5 There exist the following three equilibria corresponding to system (5.10) namely, (i) $E_1(0,0,0,0)$, (ii) $E_2(0,0,K_1,0)$ and (iii) $E_3(Y_1^*, Z_1^*, N_1^*, Y_2^*)$, which exists if

$$\frac{\beta_1\left\{\beta_2 + \lambda_2 \dfrac{\delta_1}{d_1 + (1-a)r_1}\right\}\bar{N}_2 K_1}{\left\{\alpha_2 + d_2 + (1-a')\dfrac{r_2}{K_2}\bar{N}_2\right\}\{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)r_1\}} = \bar{R}_s > 1.$$

This equilibrium is unique if $\frac{dY_1}{dN_1} \geq 0$ for all $N_1 > 0$, and when $N_{10}^* > \frac{K}{2}$, where Y_1 is given in equation (5.15) of the proof of this theorem.

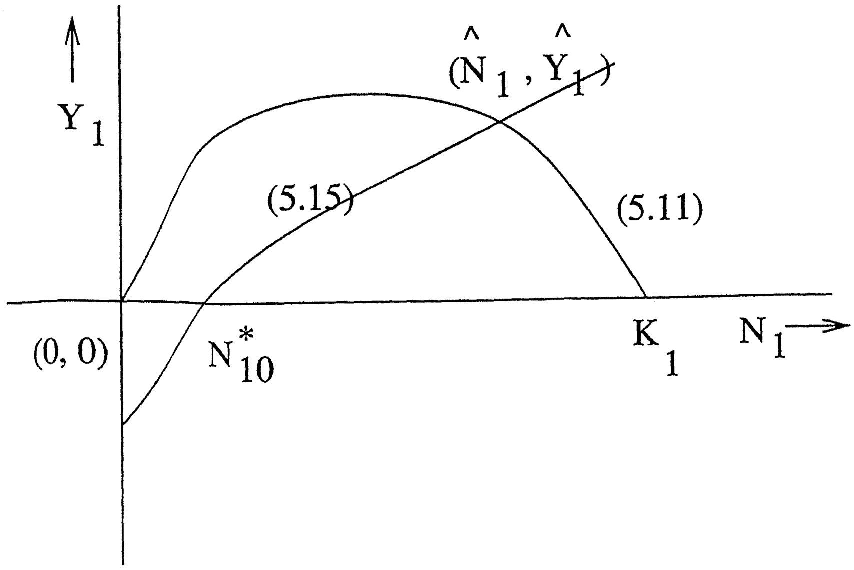


Figure 5.10: Existence of equilibrium point.

Proof: The existence of the first two equilibria is obvious. We prove the existence of the third nontrivial equilibrium by the isocline method as follows :
Setting the right hand sides of (5.10) to zero, we get

$$Y_1 = \frac{r_1}{\alpha_1}(1 - \frac{N_1}{K_1})N_1, \tag{5.11}$$

$$Z_1 = \frac{\delta_1}{d_1 + (1-a)\frac{r_1}{K_1}N_1}Y_1, \tag{5.12}$$

$$Y_2 = \frac{\{\beta_2 + \lambda_2 \frac{\delta_1}{d_1+(1-a)\frac{r_1}{K_1}N_1}\}\bar{N}_2 Y_1}{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2 + \{\beta_2 + \lambda_2 \frac{\delta_1}{d_1+(1-a)\frac{r_1}{K_1}}\}Y_1}, \tag{5.13}$$

$$Y_2 = \frac{\{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)\frac{r_1}{K_1}N_1\}Y_1}{\beta_1 \left[N_1 - \{1 + \frac{\delta_1}{d_1+(1-a)\frac{r_1}{K_1}N_1}\}Y_1\right]}. \tag{5.14}$$

From (5.13) and (5.14), we get (for $Y_1 \neq 0$) the following:

$$Y_1 = \frac{\beta_1\beta_2^*\bar{N}_2N_1 - \{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2\}\{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)\frac{r_1}{K_1}N_1\}}{\beta_2^*[\beta_1\{1 + \frac{\delta_1}{d_1 + (1-a)\frac{r_1}{K_1}N_1}\}\bar{N}_2 + \{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)\frac{r_1}{K_1}N_1\}]}, \tag{5.15}$$

where $\beta_2^* = \beta_2 + \lambda_2 \dfrac{\delta_1}{d_1 + (1-a)\frac{r_1}{K_1}N_1}$.
We note from (5.15) that for a given N_1, we have only one value of Y_1.

Clearly in the N-Y plane (5.11) is a parabola with vertex at $\left(\frac{K_1}{2}, \frac{r_1 K_1}{4\alpha_1}\right)$ and passing through (0, 0) and $(K, 0)$.

From (5.15), the following points are noted.

(i) For $N_1 = 0,\ Y_1 = -\dfrac{(\nu_1+\alpha_1+d_1+\delta_1)\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\right\}}{(\beta_2+\lambda_2\frac{\delta_1}{d_1})\left\{\beta_1(1+\frac{\delta_1}{d_1})\bar{N}_2+\nu_1+\alpha_1+d_1+\delta_1\right\}} < 0.$

(ii) For $N_1 = K_1$,

$$Y_1 = \frac{\beta_1\beta_2^{**}\bar{N}_2K_1 - \left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\{\nu_1+\alpha_1+d_1+\delta_1+(1-a)r_1\}}{\beta_2^{**}\left[\beta_1\left\{1+\frac{\delta_1}{d_1+(1-a)r_1}\right\}\bar{N}_2+\{\nu_1+\alpha_1+d_1+\delta_1+(1-a)r_1\}\right]} > 0,$$

provided

$$\frac{\beta_1\{\beta_2+\lambda_2\frac{\delta_1}{d_1+(1-a)r_1}\}\bar{N}_2K_1}{\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\{\nu_1+\alpha_1+d_1+\delta_1+(1-a)r_1\}} = \bar{R}_s > 1, \qquad (5.16a)$$

where $\beta_2^{**} = \{\beta_2+\lambda_2\frac{\delta_1}{d_1+(1-a)r_1}\}$.

(iii) Also $Y_1 = 0$, gives the following quadratic in N_1,

$$f_1N_1^2 + f_2N_1 + f_3 = 0, \qquad (5.16b)$$

where

$$f_1 = (1-a)\frac{r_1}{K_1}\left[\beta_1\beta_2\bar{N}_2 - \left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\right\}(1-a)\frac{r_1}{K_1}\right] > 0. \qquad (5.16c)$$

The condition (5.16c) is true as we assume that (5.16a) holds for $\delta_1 = 0$.

$$f_2 = \beta_1(\beta_2d_1+\lambda_2\delta_1)\bar{N}_2 - \left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\right\}(\nu_1+\alpha_1+2d_1+\delta_1)(1-a)\frac{r_1}{K_1}$$

and

$$f_3 = -\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\right\}(\nu_1+\alpha_1+d_1+\delta_1)d_1 < 0.$$

Clearly the quadratic (5.16b) has one negative root and one positive root, say N_{10}^*.

(iv) The slope of (5.15) at $(N_{10}^*, 0)$ is

$$\left(\frac{dY_1}{dN_1}\right) = \frac{[\beta_1\beta_2\bar{N}_2 - \{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\}(1-a)\frac{r_1}{K_1}]\ D^{*2} + \beta_1\lambda_2\delta_1d_1\bar{N}_2}{[\beta_2\ D^* + \lambda_2\delta_1]\left[\beta_1\ (D^*+\delta_1)\bar{N}_2 + \{\nu_1+\alpha_1+d_1+\delta_1+(1-a)\frac{r_1}{K_1}N_{10}^*\}\ D^*\right]} > 0$$

under condition (5.16a), where $D^* = \{d_1 + (1-a)\frac{r_1}{K_1}N_{10}^*\}$.

Now plotting (5.15) and (5.11) in the N-Y plane (Fig. 5.10), we get an intersecting point (N_1^*, Y_1^*) and then using (5.12) and (5.13), Z_1^* and Y_2^* can be determined respectively. If the slope $\frac{dY_1}{dN_1}$ of (5.15) is positive, then there exists a unique (N_1^*, Y_1^*) when $N_{10}^* > \frac{K}{2}$ and corresponding to this, a unique positive equilibrium point E_3 exists.

5.3.1.1 Stability Analysis

In the following we present the linear stability results of these equilibria.

THEOREM 5.6 The equilibrium E_1 is unstable, the equilibrium E_2 is stable if $\bar{R}_s < 1$ and unstable if $\bar{R}_s > 1$ and the equilibrium E_3 is locally asymptotically stable if $a_0 > 0$, $\begin{vmatrix} a_3 & a_1 \\ 1 & a_2 \end{vmatrix} > 0$, $\begin{vmatrix} a_3 & a_1 & 0 \\ 1 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0$, where a_0, a_1, a_2, and a_3 are given explicitly in the proof of the theorem.

Proof: The variational matrix M corresponding to the system of equations (5.10) at (Y_1, Z_1, N_1, Y_2) is given by

$$M = \begin{pmatrix} m_{11} & -\beta_1 Y_2 & \beta_1 Y_2 - (1-a)\frac{r_1}{K_1}Y_1 & \beta_1(N_1 - Y_1 - Z_1) \\ \delta_1 & -\{d_1 + (1-a)\frac{r_1}{K_1}N_1\} & -(1-a)\frac{r_1}{K_1}Z_1 & 0 \\ -\alpha_1 & 0 & r_1 - \frac{2r_1}{K_1}N_1 & 0 \\ \beta_2(\bar{N}_2 - Y_2) & \lambda_2(\bar{N}_2 - Y_2) & 0 & m_{44} \end{pmatrix},$$

where $m_{11} = -\{\beta_1 Y_2 + \nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)\frac{r_1}{K_1}N_1\}$
and $m_{44} = -[\beta_2 Y_1 + \lambda_2 Z_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2]$.

It is easy to see that the variational matrices corresponding to $E_1(0,0,0,0)$ has one positive eigenvalue implying instability of this equilibrium.

The variational matrix corresponding to E_2 has characteristic equation

$$\psi^3 + A_1\psi^2 + A_2\psi + A_3 = 0,$$

where

$$A_1 = \nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)r_1 + d_1 + (1-a)r_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N}_2,$$

$$\begin{aligned}
A_2 &= \{\nu_1+\alpha_1+d_1+\delta_1+(1-a)r_1\}\{d_1+(1-a)r_1\} \\
&\quad +\{d_1+(1-a)r_1\}\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\right\}-\beta_1\beta_2K_1\bar{N}_2, \text{ and} \\
A_3 &= \{\nu_1+\alpha_1+d_1+\delta_1+(1-a)r_1\}\{d_1+(1-a)r_1\}\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\right\} \\
&\quad -\beta_1K_1\delta_1\lambda_2\bar{N}_2-\beta_1\beta_2K_1\bar{N}_2\{d_1+(1-a)r_1\}.
\end{aligned}$$

From above equation it is easy to see that $\bar{E}_2$ is stable if $\bar{R}_s<1$ and unstable if $\bar{R}_s>1$. The variational matrix M_3 at equilibrium point $E_3(Y_1^*,Z_1^*,N_1^*,Y_2^*)$ is given by

$$M_3=\begin{pmatrix} m_{11}^* & -\beta_1Y_2^* & \beta_1Y_2^*-(1-a)\frac{r_1}{K_1}Y_1^* & \beta_1(N_1^*-Y_1^*-Z_1^*) \\ \delta_1 & m_{22}^* & -(1-a)\frac{r_1}{K_1}Z_1^* & 0 \\ -\alpha_1 & 0 & m_{33}^* & 0 \\ \beta_2(\bar{N}_2-Y_2^*) & \lambda_2(\bar{N}_2-Y_2^*) & 0 & m_{44}^* \end{pmatrix},$$

where $m_{11}^*=-\left\{\beta_1Y_2^*+\nu_1+\alpha_1+d_1+\delta_1+(1-a)r_1\frac{N_1^*}{K_1}\right\}$, $m_{22}^*=-\left\{d_1+(1-a)\frac{r_1}{K_1}N_1^*\right\}$, $m_{33}^*=-\frac{r_1}{K_1}(2N_1^*-K_1)$, and $m_{44}^*=-\left[\beta_2Y_1^*+\lambda_2Z_1^*+\left\{\alpha_2+d_2+(1-a')\frac{r_2}{K_2}\bar{N}_2\right\}\right]$.
The characteristic polynomial corresponding to above is given by

$$\psi^4+a_3\psi^3+a_2\psi^2+a_1\psi+a_0=0,$$

where

$$\begin{aligned}
a_3 &= -\{m_{11}^*+m_{22}^*+m_{33}^*+m_{44}^*\}>0, \\
a_2 &= m_{11}^*\,(m_{22}^*+m_{33}^*+m_{44}^*)+m_{22}^*\,(m_{33}^*+m_{44}^*)++m_{33}^*\,m_{44}^* \\
&\quad +\delta_1\beta_1Y_2^*+\alpha_1\left\{\beta_1Y_2^*-(1-a)\frac{r_1}{K_1}Y_1^*\right\}-\beta_1(N_1^*-Y_1^*-Z_1^*)\beta_2(\bar{N}_2-Y_2^*), \\
a_1 &= -[m_{11}^*\,m_{22}^*\,m_{33}^*+m_{22}^*\,m_{33}^*\,m_{44}^*+m_{33}^*\,m_{44}^*\,m_{11}^*+m_{44}^*\,m_{11}^*\,m_{22}^*] \\
&\quad +\beta_1Y_2^*\alpha_1(1-a)\frac{r_1}{K_1}Z_1^*-(m_{33}^*+m_{44}^*)\delta_1\beta_1Y_2^* \\
&\quad -\alpha_1\left(\beta_1Y_2^*-(1-a)\frac{r_1}{K_1}Y_1^*\right)[m_{22}^*+m_{44}^*]-\delta_1\beta_1\lambda_2(N_1^*-Y_1^*-Z_1^*)(\bar{N}_2-Y_2^*) \\
&\quad +(m_{22}^*+m_{33}^*)\beta_1(N_1^*-Y_1^*-Z_1^*)\beta_2(\bar{N}_2-Y_2^*), \\
a_0 &= m_{11}^*\,m_{22}^*\,m_{33}^*\,m_{44}^*-m_{44}^*\,\beta_1Y_2^*\alpha_1(1-a)\frac{r_1}{K_1}Z_1^* \\
&\quad +\delta_1\beta_1Y_2^*\,m_{33}^*\,m_{44}^*-\beta_1(N_1^*-Y_1^*-Z_1^*)\beta_2(\bar{N}_2-Y_2^*)\,m_{22}^*\,m_{33}^* \\
&\quad +\left\{\beta_1Y_2^*-(1-a)\frac{r_1}{K_1}Y_1^*\right\}\alpha_1\,m_{22}^*\,m_{44}^*+\delta_1\beta_1\lambda_2\,m_{33}^*(N_1^*-Y_1^*-Z_1^*)(\bar{N}_2-Y_2^*) \\
&\quad -\beta_1(N_1^*-Y_1^*-Z_1^*)(1-a)\frac{r_1}{K_1}Z_1^*\alpha_1\lambda_2(\bar{N}_2-Y_2^*).
\end{aligned}$$

By the Routh-Hurwitz criteria, conditions for the local stability of the system are the following:

$$a_3 > 0, \quad \begin{vmatrix} a_3 & a_1 \\ 1 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_3 & a_1 & 0 \\ 1 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0, \quad \begin{vmatrix} a_3 & a_1 & 0 & 0 \\ 1 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & 1 & a_2 & a_0 \end{vmatrix} > 0.$$

Remark: It is noted that $a_3 > 0$ and the second inequality is obvious if $N_1^* > \frac{K_1}{2}$. So if the third inequality is satisfied then the fourth one is obvious provided $a_0 > 0$. Thus, under these conditions this equilibrium point is locally asymptotically stable.

Nonlinear Analysis and Simulation: As before, here also we speculate that the equilibrium E_3 is globally stable under local stability conditions.
To illustrate this, the system (5.10) is integrated by the fourth order Runge-Kutta method using the following parameter values in the simulation.

$$\beta_1 = 0.00000022 = \beta_2, \ \nu_1 = 0.012, \ \alpha_1 = 0.0005, \ d_1 = 0.0004, \ \delta_1 = 0.00002,$$

$$a = 0.3, \ r_1 = 0.0003, \ K_1 = 50000, \ \lambda_2 = 0.00000011, \ \alpha_2 = 0.045, \ a' = 0.999,$$

$$K_2 = 1000000, \ \ r_2 = 1.0, \ d_2 = 0.04, \ \alpha_2 = 0.045, \ \delta_2 = 0.0000002, \ Q_a = 20, \ \delta_0 = 0.001.$$

The equilibrium values of $\hat{Y}_1, \ \hat{Z}_1, \ \hat{N}_1$ and $\hat{Y}_2$ are as follows

$$\hat{Y}_1 = 6986.664, \quad \hat{Z}_1 = 262.425, \quad \hat{N}_1 = 31540.491, \quad \hat{Y}_2 = 17064.251.$$

Simulation is performed for different initial positions 1, 2, 3 and 4 shown in Fig. 5.11. In this figure, the infected human population is plotted against the susceptible population and also against total population N. From the solution curves, we conclude that the system is globally stable about this equilibrium point under above conditions provided we start away from other equilibria. Also in Figures 5.12 and 5.13, the infected population is plotted against time for different r_1, Q_a, δ_1 and it is noted that with the increase of any of these parameters, the infective population increases showing that the disease spreads faster due to these effects.

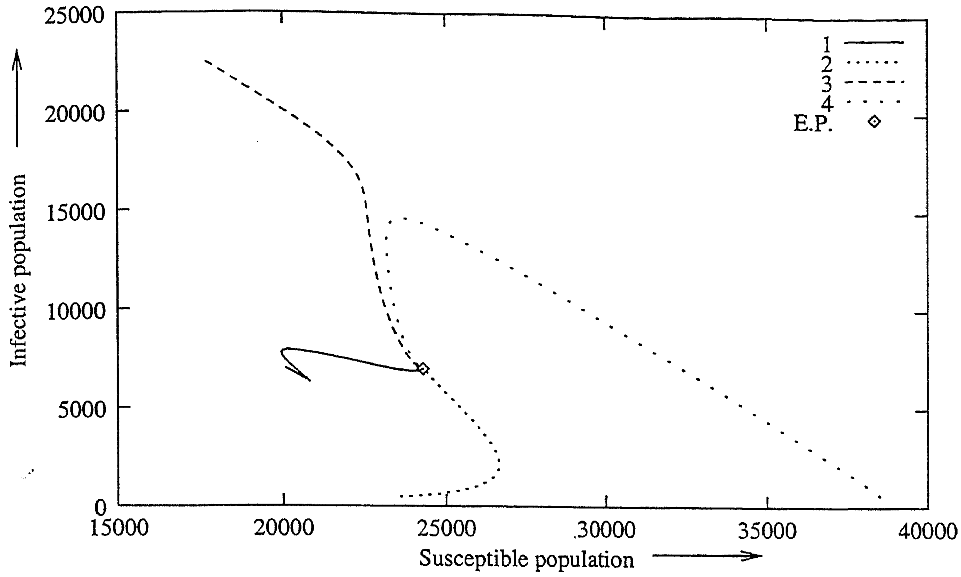


Figure 5.11: Variation of infective human population with human susceptible population.

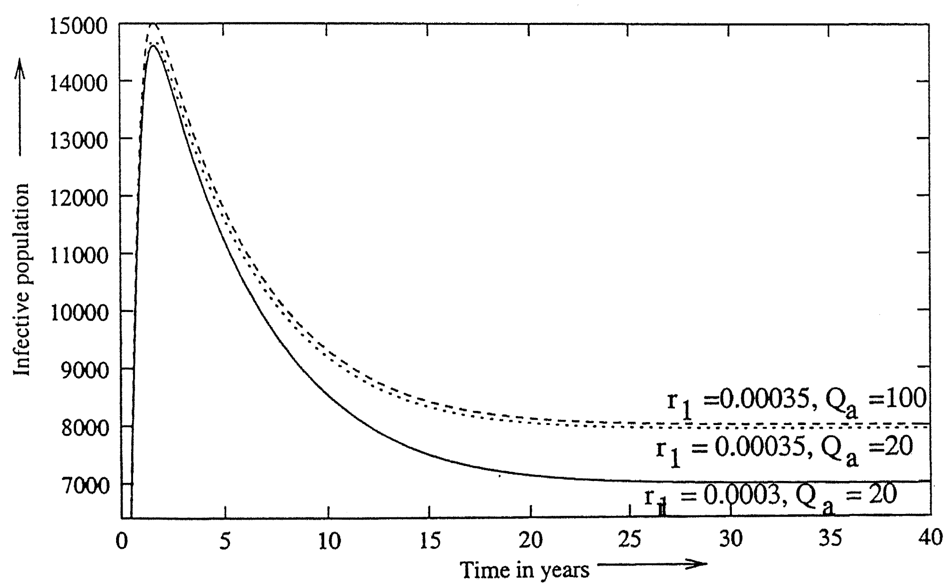


Figure 5.12: Variation of infective human population with time for different intrinsic growth rates of human population and different rates of cumulative environmental discharges.

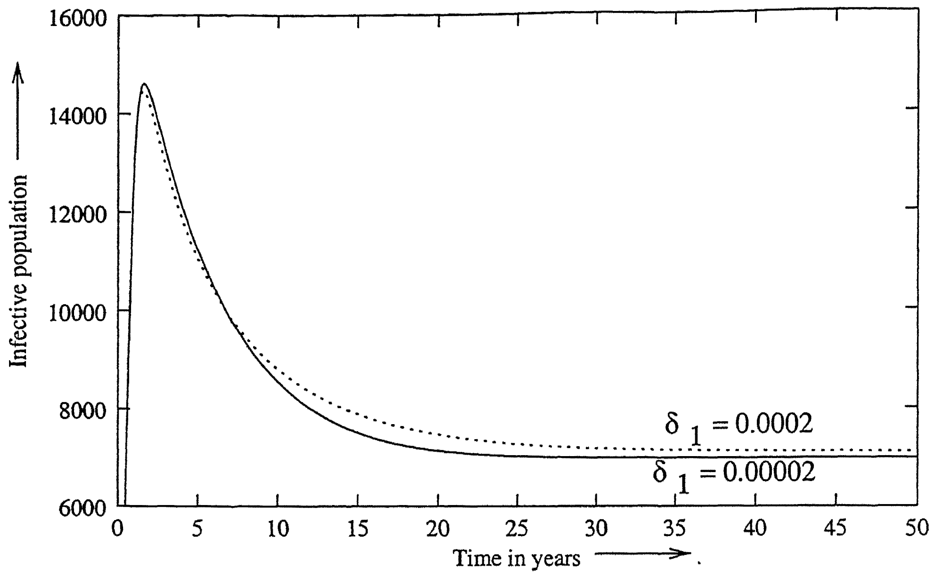


Figure 5.13: Variation of infective population with time for different rate coefficients corresponding to movement of human population from infective class to reservoir class.

5.3.2 Case II: $Q = Q_0 + lN_1$

As indicated before, in this case it is sufficient to consider the following equivalent system of the system of equations (5.9).

$$
\begin{aligned}
\dot{Y_1} &= \beta_1(N_1 - Y_1 - Z_1)Y_2 - \left\{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)\frac{r_1}{K_1}N_1\right\}Y_1, \\
\dot{Z_1} &= \delta_1 Y_1 - \left\{d_1 + (1-a)\frac{r_1}{K_1}N_1\right\}Z_1, \\
\dot{N_1} &= r_1\left(1 - \frac{N_1}{K_1}\right)N_1 - \alpha_1 Y_1, \\
\dot{Y_2} &= \beta_2(N_2 - Y_2)Y_1 + \lambda_2(N_2 - Y_2)Z_1 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2\right\}Y_2, \\
\dot{N_2} &= r_2N_2\left(1 - \frac{N_2}{K_2}\right) - \alpha_2N_2 + \delta_2N_2E, \\
\dot{E} &= Q_0 + lN_1 - \delta_0E.
\end{aligned}
\tag{5.17}
$$

The result of an equilibrium analysis is stated in the following theorem.

THEOREM 5.7 There exist the following five equilibria, namely
(i) $E_1\left(0,0,0,0,0,\frac{Q_0}{\delta_0}\right)$, *(ii)* $E_2\left(0,0,0,\frac{K_2}{r_2}\left\{r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0}\right\},\frac{Q_0}{\delta_0}\right)$,

(iii) $E_3\left(0,0,K_1,0,0,\frac{Q_0+lK_1}{\delta_0}\right)$, *(iv)* $E_4\left(0,0,K_1,0,N_2^*,\frac{Q_0+lK_1}{\delta_0}\right)$,

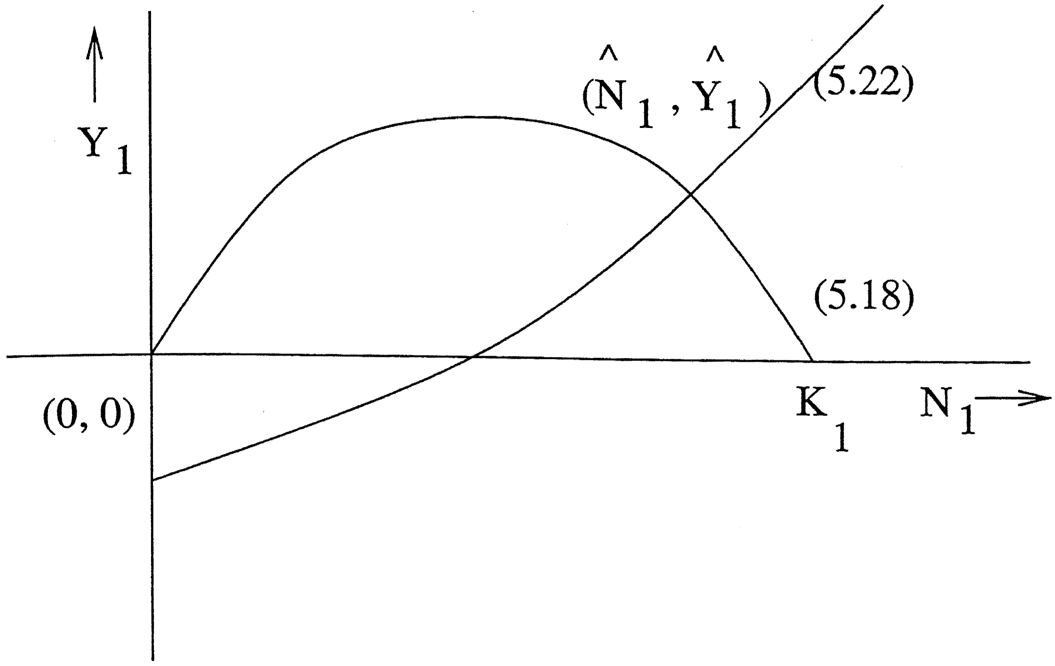


Figure 5.14: Existence of equilibrium point.

where $N_2^* = \dfrac{K_2}{r_2}\left\{r_2 - \alpha_2 + \dfrac{\delta_2}{\delta_0}(Q_0 + lK_1)\right\}$,

and (v) $E_5(\hat{Y}_1, \hat{Z}_1, \hat{N}_1, \hat{Y}_2, \hat{N}_2, \hat{E})$. E_5 *exists if*

$$\frac{\beta_1\left\{\beta_2 + \lambda_2 \dfrac{\delta_1}{d_1 + (1-a)r_1}\right\} N_2^* K_1}{\left\{\alpha_2 + d_2 + (1-a')\dfrac{r_2}{K_2}N_2^*\right\}\{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)r_1\}} = \bar{\bar{R}}_s \ > \ 1.$$

This equilibrium is unique if $\frac{dY_1}{dN_1} \geq 0$ *and* $N_{10}* > \frac{K_1}{2}$, *where* Y_1 *and* N_{10}^* *are given in the proof of the theorem.*

Proof: The existence of the first four equilibria is obvious. The existence of E_5 is shown by the isocline method. Setting the right hand sides of (5.17) to zero, we have the following equations, when $N_1 \neq 0, \quad N_1 \neq K_1, \quad N_2 \neq 0$,

$$Y_1 = \frac{r_1}{\alpha_1}\left(1 - \frac{N_1}{K_1}\right) N_1, \tag{5.18}$$

$$E = \frac{Q_0 + lN_1}{\delta_0}, \quad Z_1 = \frac{\delta_1}{d_1 + (1-a)\frac{r_1}{K_1}N_1} Y_1, \quad N_2 = \frac{K_2}{r_2}\{r_2 - \alpha_2 + \frac{\delta_2}{\delta_0}(Q_0 + lN_1)\}, \tag{5.19}$$

$$Y_2 = \frac{\left\{\beta_2 + \lambda_2 \frac{\delta_1}{d_1+(1-a)\frac{r_1}{K_1}N_1}\right\} Y_1 N_2}{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2 + \left\{\beta_2 + \lambda_2 \frac{\delta_1}{d_1+(1-a)\frac{r_1}{K_1}N_1}\right\} Y_1}, \tag{5.20}$$

$$Y_2 = \frac{\left\{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)\frac{r_1}{K_1}N_1\right\} Y_1}{\beta_1 \left[N_1 - \left\{1 + \frac{\delta_1}{d_1+(1-a)\frac{r_1}{K_1}N_1}\right\} Y_1\right]}. \tag{5.21}$$

From (5.20) and (5.21), we get

$$Y_1 = \frac{\beta_1\beta_2^* N_2 N_1 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2\right\}\left\{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)\frac{r_1}{K_1}N_1\right\}}{\beta_2^*\left[\beta_1\left\{1 + \frac{\delta_1}{d_1 + (1-a)\frac{r_1}{K_1}N_1}\right\} N_2 + \left\{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)\frac{r_1}{K_1}N_1\right\}\right]} \tag{5.22},$$

where $N_2 = \dfrac{K_2}{r_2}\left\{r_2 - \alpha_2 + \dfrac{\delta_2}{\delta_0}(Q_0 + lN_1)\right\}$.

Clearly in the N-Y plane (5.18) is a parabola with vertex at $\left(\frac{K_1}{2}, \frac{r_1K_1}{4\alpha_1}\right)$ and passing through (0, 0) and $(K, 0)$.

From (5.22), the following points are observed.

(i) For $N_1 = 0$,

$$Y_1 = -\frac{(\nu_1 + \alpha_1 + d_1 + \delta_1)\left\{\alpha_2 + d_2 + (1-a')\left(r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0}\right)\right\}}{\left(\beta_2 + \lambda_2\frac{\delta_1}{d_1}\right)\left\{\beta_1\left(1 + \frac{\delta_1}{d_1}\right)\frac{K_2}{r_2}\left(r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0}\right) + \nu_1 + \alpha_1 + d_1 + \delta_1\right\}} < 0.$$

(ii) For $N_1 = K_1$,

$$Y_1 = \frac{\beta_1\beta_2^{**}N_2^*K_1 - \{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\}\{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)\frac{r_1}{K_1}K_1\}}{\beta_2^{**}[\beta_1\{1 + \frac{\delta_1}{d_1 + (1-a)r_1}\}N_2^* + \{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)r_1\}]} > 0,$$

provided

$$\frac{\beta_1\beta_2^{**}N_2^*K_1}{\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\}\{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)r_1\}} = \bar{\bar{R}}_s > 1, \tag{5.23}$$

where $\beta_2^{**} = \beta_2 + \lambda_2\frac{\delta_1}{d_1+(1-a)r_1}$.

(iii) For $Y_1 = 0$, we get the following cubic in N_1

$$h_1N_1^3 + h_2N_1^2 + h_3N_1 + h_4 = 0, \tag{5.24}$$

where

$$
\begin{aligned}
h_1 &= (1-a)\frac{r_1}{K_1}\frac{\delta_2 l}{\delta_0}\left\{\beta_1\beta_2\frac{K_2}{r_2} - (1-a)(1-a')\frac{r_1}{K_1}\right\},\\
h_2 &= \beta_1\left[(\beta_2 d_1 + \lambda_2\delta_1)\frac{K_2\delta_2 l}{\delta_0} + \frac{K_2}{r_2}\left(r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0}\right)\beta_2(1-a)\frac{r_1}{K_1}\right] - \frac{(1-a)r_1}{K_1}\left[\frac{d_1(1-a')\delta_2 l}{\delta_0}\right.\\
&\left.+\left\{\alpha_2 + d_2 + (1-a')\left(r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0}\right)(1-a)\frac{r_1}{K_1} + (\nu_1 + \alpha_1 + d_1 + \delta_1)(1-a')\frac{\delta_2 l}{\delta_0}\right\}\right],\\
h_3 &= \beta_1(\beta_2 d_1 + \lambda_2\delta_1)\frac{K_2}{r_2}\left\{r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0}\right\} - d_1\left[(1-a')\frac{\delta_2 l}{\delta_0}(\nu_1 + \alpha_1 + d_1 + \delta_1)\right.\\
&\left.+(1-a)\frac{r_1}{K_1}\left\{\alpha_2 + d_2 + (1-a')(r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0})\right\}\right]\\
&-(1-a)\frac{r_1}{K_1}\left\{\alpha_2 + d_2 + (1-a')\left(r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0}\right)\right\}(\nu_1 + \alpha_1 + d_1 + \delta_1),\\
h_4 &= -d_1\left\{\alpha_2 + d_2 + (1-a')(r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0})\right\}(\nu_1 + \alpha_1 + d_1 + \delta_1).
\end{aligned}
$$

It is noted that under condition (5.23), the coefficient of N_1^3 i.e. h_1 is positive and the constant term h_4 is negative in (5.24), thus implying that it has one positive root. If the slope $\frac{dY_1}{dN_1}$ is positive for $N_1 > 0$, then under the above condition plotting (5.18) and (5.22) in the N-Y plane (Fig. 5.14), we get a unique intersecting point as $(\hat{Y}_1, \hat{N}_1)$ provided $N_{10}^* \geq \frac{K_1}{2}$. $\hat{Y}_2$, $\hat{N}_2$, $\hat{Z}_1$ and $\hat{E}$ are then easily derived by using (5.19) and (5.20).

5.3.2.1 Stability Analysis

The linear stability results are stated in the following theorem.

THEOREM 5.8 The equilibria E_1, E_2 and E_3 are unstable, the equilibrium E_4 is locally asymptotically stable if $\bar{\bar{R}}_s < 1$, if $\bar{\bar{R}}_s > 1$ it is unstable and the equilibrium E_5 exists, which is locally asymptotically stable provided

$$
a_0 > 0, \begin{vmatrix} a_5 & a_3 \\ 1 & a_4 \end{vmatrix} > 0, \begin{vmatrix} a_5 & a_3 & a_1 \\ 1 & a_4 & a_2 \\ 0 & a_5 & a_3 \end{vmatrix} > 0, \begin{vmatrix} a_5 & a_3 & a_1 & 0 \\ 1 & a_4 & a_2 & a_0 \\ 0 & a_5 & a_3 & a_1 \\ 0 & 1 & a_4 & a_2 \end{vmatrix} > 0, \begin{vmatrix} a_5 & a_3 & a_1 & 0 & 0 \\ 1 & a_4 & a_2 & a_0 & 0 \\ 0 & a_5 & a_3 & a_1 & 0 \\ 0 & 1 & a_4 & a_2 & a_0 \\ 0 & 0 & a_5 & a_3 & a_1 \end{vmatrix} > 0,
$$

where a_0, a_1, a_2, a_3, a_4 and a_5 are given explicitly in the proof of the theorem.

Proof: The variational matrix M at the equilibrium point $(Y_1, Z_1, N_1, Y_2, N_2, E)$ is given by

$$M = \begin{pmatrix} m_{11} & -\beta_1 Y_2 & m_{13} & m_{14} & 0 & 0 \\ \delta_1 & -\{d_1 + (1-a)\frac{r_1}{K_1}N_1\} & -(1-a)\frac{r_1}{K_1}Z_1 & 0 & 0 & 0 \\ -\alpha_1 & 0 & r_1 - \frac{2r_1}{K_1}N_1 & 0 & 0 & 0 \\ \beta_2(N_2 - Y_2) & \lambda_2(N_2 - Y_2) & 0 & m_{44} & m_{45} & 0 \\ 0 & 0 & 0 & 0 & m_{55} & \delta_2 N_2 \\ 0 & 0 & l & 0 & 0 & -\delta_0 \end{pmatrix},$$

where $m_{11} = -(\beta_1 Y_2 + \nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)\frac{r_1}{K_1}N_1)$, $m_{13} = \beta_1 Y_2 - (1-a)\frac{r_1}{K_1}Y_1$, $m_{14} = \beta_1(N_1 - Y_1 - Z_1)$, $m_{44} = -\{\beta_2 Y_1 + \lambda_2 Z_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}Y_2\}$, $m_{45} = \beta_2 Y_1 + \lambda_2 Z_1 - (1-a')\frac{r_2}{K_2}Y_2$, $m_{55} = r_2 - \alpha_2 + \delta_2 E - \frac{2r_2}{K_2}N_2$.

It is easy to see that the variational matrices corresponding to $E_1\left(0,0,0,0,0,\frac{Q_0}{\delta_0}\right)$, $E_2\left(0,0,0,0,\frac{K_2}{r_2}\{r_2 - \alpha_2 + \delta_2\frac{Q_0}{\delta_0}\},\frac{Q_0}{\delta_0}\right)$ and $E_3(0,0,K_1,0,0,\frac{Q_0+lK_1}{\delta_0})$ have one positive characteristic root, thus implying instability of these equilibria.

The variational matrix M_4 corresponding to the equilibrium point E_4 is given by

$$M_4 = \begin{pmatrix} m_{11} & 0 & 0 & \beta_1 K_1 & 0 & 0 \\ \delta_1 & -\{d_1 + (1-a)r_1\} & 0 & 0 & 0 & 0 \\ -\alpha_1 & 0 & -r_1 & 0 & 0 & 0 \\ \beta_2 N_2^* & \lambda_2 N_2^* & 0 & m_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{55} & \delta_2 N_2^* \\ 0 & 0 & l & 0 & 0 & -\delta_0 \end{pmatrix},$$

where $m_{11} = -\{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)r_1\}$, $m_{44} = -\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\}$ and $m_{55} = -\{r_2 - \alpha_2 + \frac{\delta_2}{\delta_0}(Q_0 + lK_1)\}$.

Clearly three roots of the above matrix are $-\delta_0$, $-r_1$, $-\{r_2 - \alpha_2 + \frac{\delta_2}{\delta_0}(Q_0 + lK_1)\}$ and the other roots are given by following cubic equation

$$\psi^3 + a_1\psi^2 + a_2\psi + a_3 = 0,$$

where

$$\begin{aligned} a_1 &= \nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)r_1 + d_1 + (1-a)r_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^* > 0, \\ a_2 &= \{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)r_1\}\left\{d_1 + (1-a)r_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\right\} \\ &\quad \{d_1 + (1-a)r_1\}\left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\right\} - \beta_1\beta_2 K_1 N_2^*, \\ a_3 &= \{d_1 + (1-a)r_1\}\{\nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)r_1\}\left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2^*\right\} \\ &\quad -\beta_1 K_1\left\{\beta_2(d_1 + \overline{1-a}r_1) + \delta_1\lambda_2\right\}N_2^*. \end{aligned}$$

By the Routh-Hurwitz criteria the equilibrium E_4 is locally asymptotically stable if a_2, a_3 are positive and also $a_1a_2 - a_3 > 0$, which are satisfied if $\bar{\bar{R}}_s < 1$, otherwise if $\bar{\bar{R}}_s > 1$ this is unstable and the fifth equilibrium exists.

Now the variational matrix M_5 corresponding to the fifth equilibrium $E_5(\hat{Y}_1, \hat{Z}_1, \hat{N}_1, \hat{Y}_2, \hat{N}_2, \hat{E})$ is given by

$$M_5 = \begin{pmatrix} \hat{m}_{11} & -\beta_1\hat{Y}_2 & \hat{m}_{13} & \hat{m}_{14} & 0 & 0 \\ \delta_1 & \hat{m}_{22} & -(1-a)\frac{r_1}{K_1}\hat{Z}_1 & 0 & 0 & 0 \\ -\alpha_1 & 0 & \hat{m}_{33} & 0 & 0 & 0 \\ \beta_2(\hat{N}_2 - \hat{Y}_2) & \lambda_2(\hat{N}_2 - \hat{Y}_2) & 0 & \hat{m}_{44} & \hat{m}_{45} & 0 \\ 0 & 0 & 0 & 0 & \hat{m}_{55} & \delta_2\hat{N}_2 \\ 0 & 0 & l & 0 & 0 & -\delta_0 \end{pmatrix}$$

where $\hat{m}_{11} = -(\beta_1\hat{Y}_2 + \nu_1 + \alpha_1 + d_1 + \delta_1 + (1-a)\frac{r_1}{K_1}\hat{N}_1)$, $\hat{m}_{22} = -\{d_1 + (1-a)\frac{r_1}{K_1}\hat{N}_1\}$,
$\hat{m}_{13} = \beta_1\hat{Y}_2 - (1-a)\frac{r_1}{K_1}\hat{Y}_1$, $\hat{m}_{14} = \beta_1(\hat{N}_1 - \hat{Y}_1 - \hat{Z}_1)$, $\hat{m}_{33} = -\frac{r_1}{K_1}(2\hat{N}_1 - K_1)$,
$\hat{m}_{44} = -\{\beta_2\hat{Y}_1 + \lambda_2\hat{Z}_1 + \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\hat{N}_2\}$,
$\hat{m}_{45} = \beta_2\hat{Y}_1 + \lambda_2\hat{Z}_1 - (1-a')\frac{r_2}{K_2}Y_2$ and $\hat{m}_{55} = -\frac{r_2}{K_2}\hat{N}_2$.

The characteristic polynomial corresponding to the above matrix is given by

$$\psi^6 + a_5\psi^5 + a_4\psi^4 + a_3\psi^3 + a_2\psi^2 + a_1\psi + a_0 = 0,$$

where

$$\begin{aligned}
a_5 &= \delta_0 - (\hat{m}_{11} + \hat{m}_{22} + \hat{m}_{33} + \hat{m}_{44} + \hat{m}_{55}) > 0, \\
a_4 &= -\delta_0(\hat{m}_{11} + \hat{m}_{22} + \hat{m}_{33} + \hat{m}_{44} + \hat{m}_{55}) + \hat{m}_{11}\ (\hat{m}_{22} + \hat{m}_{33} + \hat{m}_{44} + \hat{m}_{55}) \\
&\quad + \hat{m}_{22}\ (\hat{m}_{33} + \hat{m}_{44} + \hat{m}_{55}) + \hat{m}_{33}\ (\hat{m}_{44} + \hat{m}_{55}) + \hat{m}_{44}\ \hat{m}_{55} + \alpha_1\hat{m}_{13} \\
&\quad + \delta_1\beta_2\hat{Y}_2 - \beta_2\hat{m}_{14}(\hat{N}_2 - \hat{Y}_2), \\
a_3 &= \delta_0\hat{m}_{11}(\hat{m}_{22} + \hat{m}_{33} + \hat{m}_{44} + \hat{m}_{55}) - \hat{m}_{11}\ \hat{m}_{22}\ (\hat{m}_{33} + \hat{m}_{44} + \hat{m}_{55}) \\
&\quad - \hat{m}_{22}\ \hat{m}_{33}\ (-\delta_0 + \hat{m}_{44} + \hat{m}_{55}) - \hat{m}_{33}\ \hat{m}_{44}\ (-\delta_0 + \hat{m}_{11} + \hat{m}_{55}) \\
&\quad - \hat{m}_{44}\ \hat{m}_{55}(-\delta_0 + \hat{m}_{11} + \hat{m}_{22}) + \hat{m}_{55}\delta_0(\hat{m}_{22} + \hat{m}_{33} + \hat{m}_{44}) \\
&\quad + \delta_0\hat{m}_{22}\ \hat{m}_{44} + \beta_1\hat{Y}_2\alpha_1(1-a)\frac{r_1}{K_1}\hat{Z}_1 + (\delta_0 - \hat{m}_{33} - \hat{m}_{44} - \hat{m}_{55})\delta_1\beta_1\hat{Y}_2 \\
&\quad + \alpha_1\hat{m}_{13}(\delta_0 - \hat{m}_{22} - \hat{m}_{44} - \hat{m}_{55}) - \hat{m}_{11}\hat{m}_{33}\hat{m}_{55} \\
&\quad - \beta_2\hat{m}_{14}(\hat{N}_2 - \hat{Y}_2)(\delta_0 - \hat{m}_{22} - \hat{m}_{33} - \hat{m}_{55}) - \delta_1\lambda_2\hat{m}_{14}\delta_0(\hat{N}_2 - \hat{Y}_2), \\
a_2 &= -\delta_0\hat{m}_{11}\ \hat{m}_{22}(\hat{m}_{33} + \hat{m}_{44} + \hat{m}_{55}) + \hat{m}_{11}\ \hat{m}_{22}\ \hat{m}_{33}\ (\hat{m}_{44} + \hat{m}_{55}) + \hat{m}_{22}\ \hat{m}_{33}\ \hat{m}_{44}
\end{aligned}$$

$$
\begin{aligned}
& \times(\hat{m_{55}}-\delta_0)+\hat{m_{33}}\,\hat{m_{44}}\,\hat{m_{55}}(-\delta_0+\hat{m_{11}})-\hat{m_{44}}\,\hat{m_{55}}\,\delta_0\,(\hat{m_{11}}+\hat{m_{22}})-\delta_0\hat{m_{33}}\,\hat{m_{55}}\\
& \times(\hat{m_{22}}+\hat{m_{44}})-\delta_0\hat{m_{11}}\,\hat{m_{33}}\,(\hat{m_{44}}+\hat{m_{55}})+(\delta_0-\hat{m_{44}}-\hat{m_{55}})\beta_1\hat{Y_2}\alpha_1(1-a)\frac{r_1}{K_1}\hat{Z_1}\\
& +\delta_1\beta_1\hat{Y_2}\{-\delta_0\,(\hat{m_{33}}+\hat{m_{44}}+\hat{m_{55}})+\hat{m_{33}}(\hat{m_{44}}+\hat{m_{55}})+\hat{m_{44}}\hat{m_{55}}\}\\
& +\alpha_1\hat{m_{13}}\{-\delta_0(\hat{m_{22}}+\hat{m_{44}}+\hat{m_{55}})+\hat{m_{22}}(\hat{m_{44}}+\hat{m_{55}})+\hat{m_{44}}\hat{m_{55}}\}\\
& -\hat{m_{14}}(\hat{N_2}-\hat{Y_2})\beta_2\,(\hat{m_{22}}\hat{m_{33}}+\hat{m_{22}}\hat{m_{55}}+\hat{m_{33}}\hat{m_{55}}-\delta_0(\hat{m_{22}}+\hat{m_{33}}+\hat{m_{55}}))\\
& -\hat{m_{14}}(\hat{N_2}-\hat{Y_2})\lambda_2\alpha_1(1-a)\frac{r_1}{K_1}\hat{Z_1}-\hat{m_{14}}(\hat{N_2}-\hat{Y_2})\delta_1\lambda_2(\delta_0-\hat{m_{33}}-\hat{m_{55}}),\\
a_1 \;=\; & \delta_0\hat{m_{11}}\,\hat{m_{22}}\,(\hat{m_{33}}\,\hat{m_{44}}+\hat{m_{44}}\,\hat{m_{55}}+\hat{m_{55}}\,\hat{m_{33}})\\
& +\hat{m_{33}}\,\hat{m_{44}}\,\hat{m_{55}}\,\{\delta_0(\hat{m_{11}}+\hat{m_{22}})-\hat{m_{11}}\,\hat{m_{22}}\}\\
& +\alpha_1(1-a)\frac{r_1}{K_1}\hat{Z_1}\beta_1\hat{Y_2}\,\{\hat{m_{44}}\hat{m_{55}}-\delta_0(\hat{m_{44}}+\hat{m_{55}})\}\\
& +\delta_1\beta_1\hat{Y_2}\{\delta_0\,\hat{m_{44}}\,\hat{m_{55}}-\,\hat{m_{44}}\,\hat{m_{55}}\,\hat{m_{33}}+\hat{m_{55}}\,\hat{m_{33}}\delta_0+\hat{m_{33}}\,\delta_0\,\hat{m_{44}}\}\\
& +\alpha_1\hat{m_{13}}\{\delta_0\hat{m_{22}}\,\hat{m_{44}}-\hat{m_{22}}\,\hat{m_{44}}\,\hat{m_{55}}+\delta_0\,\hat{m_{55}}\,(\hat{m_{22}}+\hat{m_{44}})\}\\
& +\hat{m_{14}}(\hat{N_2}-\hat{Y_2})\beta_2\,\{\hat{m_{22}}\hat{m_{33}}\hat{m_{55}}-\delta_0(\hat{m_{22}}\hat{m_{33}}+m_{33}\hat{m_{55}}+\hat{m_{22}}\hat{m_{55}})\}\\
& -\hat{m_{14}}(\hat{N_2}-\hat{Y_2})\lambda_2\alpha_1(1-a)\frac{r_1}{K_1}\hat{Z_1}(\delta_0-\hat{m_{55}})-\delta_0\hat{N_2}\hat{m_{45}}\hat{m_{14}}l\alpha_1\\
& -\delta_1\lambda_2\hat{m_{14}}(\hat{N_2}-\hat{Y_2})\{\hat{m_{33}}\hat{m_{55}}-\delta_0(\hat{m_{33}}+\hat{m_{55}})\},\\
a_0 \;=\; & \delta_0\hat{m_{44}}\,\hat{m_{55}}\left(-\hat{m_{11}}\hat{m_{22}}\hat{m_{33}}+\alpha_1\beta_1\hat{Y_2}(1-a)\frac{r_1}{K_1}\hat{Z_1}-\alpha_1\hat{m_{22}}\hat{m_{33}}-\delta_1\beta_1\hat{Y_2}m_{33}\right)\\
& +\hat{m_{14}}\hat{m_{55}}\delta_0(\hat{N_2}-\hat{Y_2})\left\{\beta_2\hat{m_{22}}\hat{m_{33}}+\lambda_2\alpha_1(1-a)\frac{r_1}{K_1}\hat{Z_1}-\delta_1\lambda_2\hat{m_{33}}\right\}\\
& +\delta_2\hat{N_2}\hat{m_{45}}\hat{m_{14}}\hat{m_{22}}l\alpha_1.
\end{aligned}
$$

By the Routh-Hurwitz criteria, if the following conditions are satisfied then this equilibrium point is stable,

$$
a_5>0,\quad \begin{vmatrix} a_5 & a_3\\ 1 & a_4\end{vmatrix}>0,\quad \begin{vmatrix} a_5 & a_3 & a_1\\ 1 & a_4 & a_2\\ 0 & a_5 & a_3\end{vmatrix}>0,\quad \begin{vmatrix} a_5 & a_3 & a_1 & 0\\ 1 & a_4 & a_2 & a_0\\ 0 & a_5 & a_3 & a_1\\ 0 & 1 & a_4 & a_2\end{vmatrix}>0,
$$

$$
\begin{vmatrix} a_5 & a_3 & a_1 & 0 & 0\\ 1 & a_4 & a_2 & a_0 & 0\\ 0 & a_5 & a_3 & a_1 & 0\\ 0 & 1 & a_4 & a_2 & a_0\\ 0 & 0 & a_5 & a_3 & a_1\end{vmatrix}>0,\quad \begin{vmatrix} a_5 & a_3 & a_1 & 0 & 0 & 0\\ 1 & a_4 & a_2 & a_0 & 0 & 0\\ 0 & a_5 & a_3 & a_1 & 0 & 0\\ 0 & 1 & a_4 & a_2 & a_0 & 0\\ 0 & 0 & a_5 & a_3 & a_1 & 0\\ 0 & 0 & 1 & a_4 & a_2 & a_0\end{vmatrix}>0.
$$

Remark: It is noted that $a_5 > 0$ and the second inequality is obvious if $\hat{N}_1 > \frac{K_1}{2}$. If next three inequalities are satisfied then so is the sixth provided $a_0 > 0$. So for numerical simulation, we choose parameters such that $\hat{N}_1 > \frac{K_1}{2}$.

Nonlinear Analysis and Simulation: Here also the global stability of E_5 is speculated under local stability conditions. To show this, the system (5.17) is integrated by the fourth order Runge-Kutta method using the same parameter values as in the previous subsection with $Q_0 = Q_a$ and an additional parameter $l = 0.00005$. The equilibrium values of $\hat{Y}_1$, $\hat{Z}_1$, $\hat{N}_1$, $\hat{Y}_2$, $\hat{N}_2$ and $\hat{E}$ are as follows:

$\hat{Y}_1 = 7014.7$, $\hat{Z}_1 = 263.9$, $\hat{N}_1 = 31359.4$, $\hat{Y}_2 = 17281.5$, $\hat{N}_2 = 962135.9$, $\hat{E} = 35679.7$.

Simulation is performed for different initial positions 1, 2, 3 and 4 in the interior of the region of attraction as shown in Fig. 5.15. In this figure, infected population is plotted against susceptible population. From the solution curves, we conclude that the system is globally stable about this equilibrium provided that we start away from other equilibria. Also from Figs. 5.16-5.20, where the infective population is plotted with time for different r_2, δ_2, Q_0, l and δ_1, we note that the infective population increases as any of these parameters increases showing the effect of these parameters on the spread of disease.

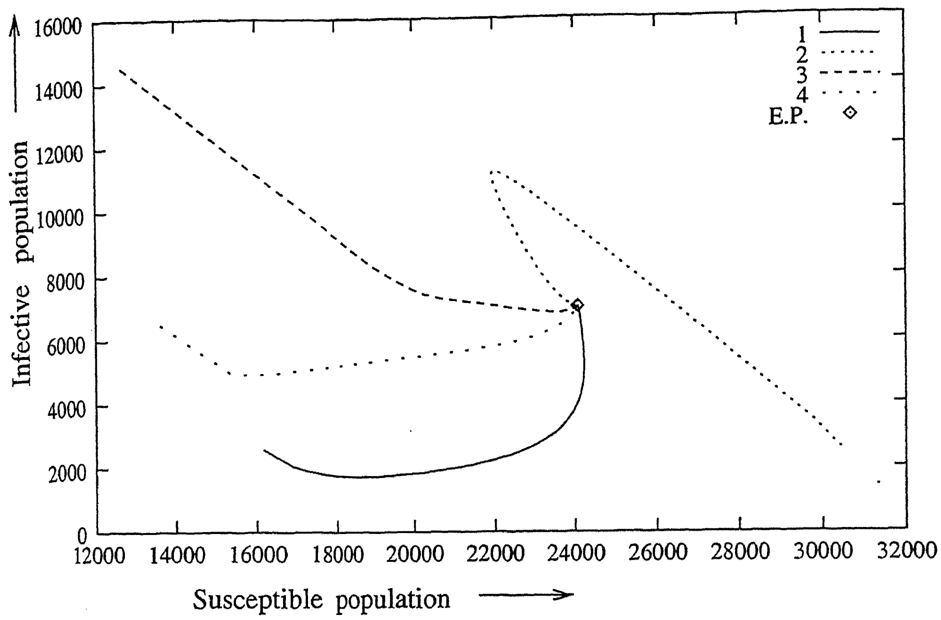


Figure 5.15: Variation of infective human population with susceptible human population.

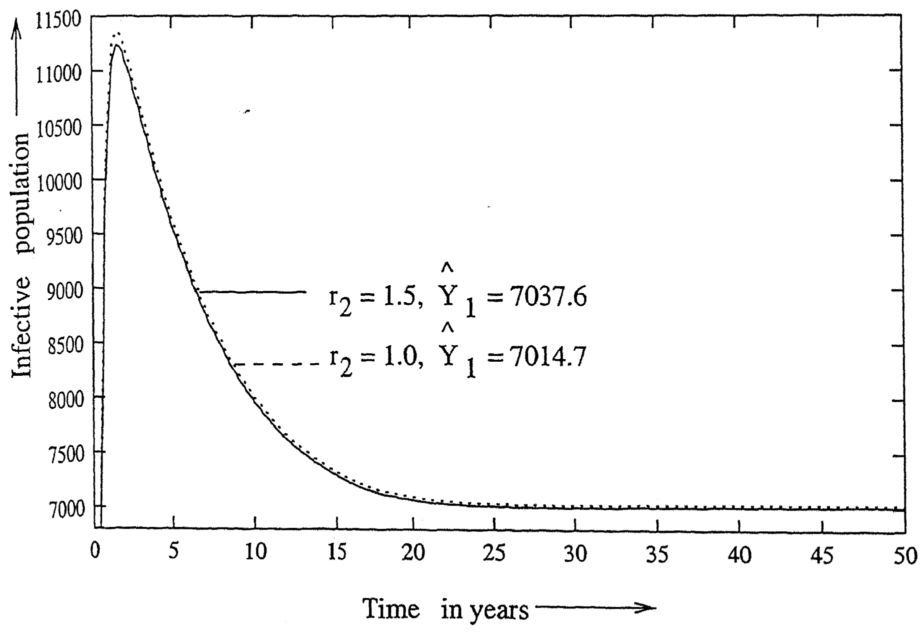


Figure 5.16: Variation of infective human population with time for different intrinsic growth rates of mosquito population.

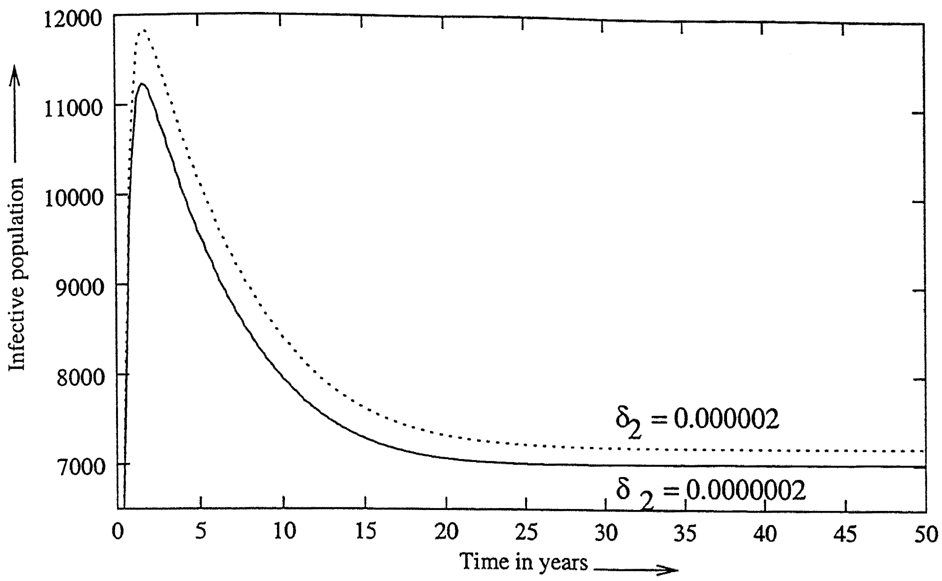


Figure 5.17: Variation of infective human population with time for different growth rate coefficients of mosquito population due to environmental discharges.

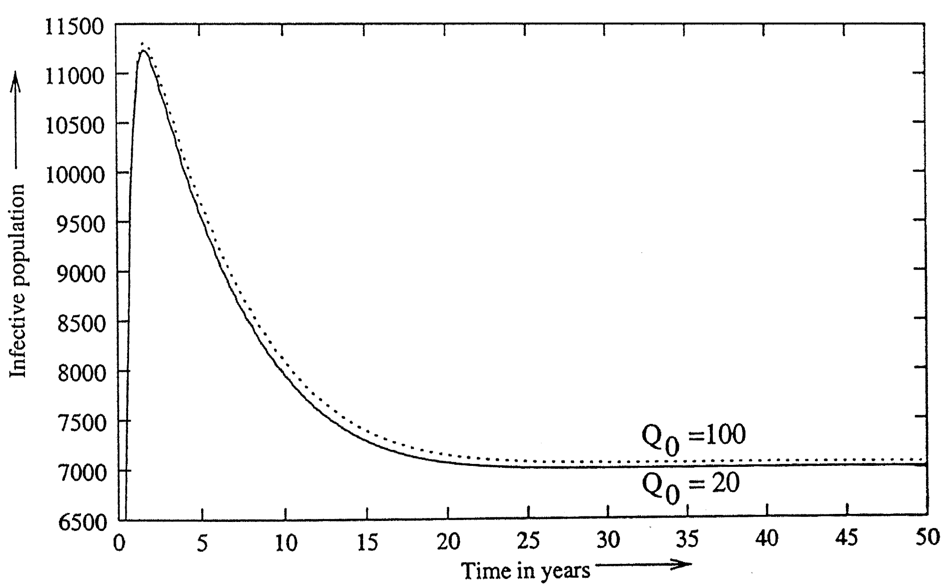


Figure 5.18: Variation of infective human population with time for different rates of cumulative environmental discharges.

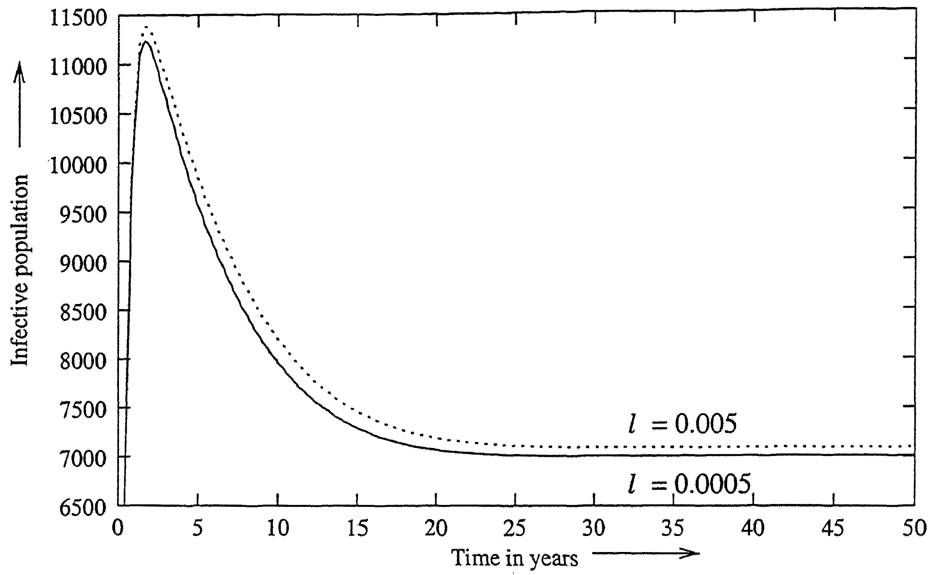


Figure 5.19: Variation of infective population with time for different l.

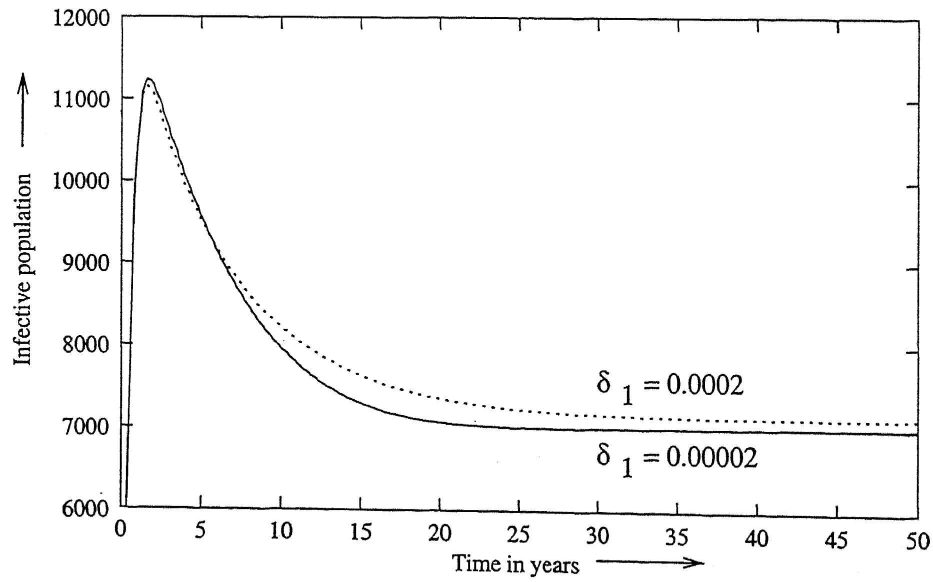


Figure 5.20: Variation of infective population with time for different rate coefficients corresponding to movement of human population from infective class to reservoir class.

5.4 Conclusions

In this chapter an SIS model for malaria with human population as reservoir is proposed and analyzed by considering environmental and demographic effects. The cases of constant rate of cumulative environmental discharges as well as a population dependent rate of cumulative environmental discharges are considered. The following two types of demographics for human population are considered, (i) constant immigration and (ii) logistic population growth. The threshold condition for spread of malaria is derived in each case. It is shown in each case that if the threshold is greater than one, the nontrivial equilibrium is always feasible and is locally asymptotically stable to small perturbations under certain conditions. If a public health measure is such that it keeps threshold less than one then disease will die out, i.e. the disease free equilibrium will be stable. By simulation it is observed that under local stability conditions the nontrivial equilibria seem to be globally stable, provided that we start away from the other equilibria. It is shown that due to environmental discharges, the mosquito population can grows very large leading to increased spread of malaria. Also if the rate of immigration or the growth rate of human population increases, the infective population increases implying further increase in the spread of malaria and the disease becomes more endemic.

Chapter 6

Modelling the Spread of a Carrier-Dependent Infectious Disease in Two Neighboring Habitats with Migration in Between

6.1 Introduction

As pointed out in Chapter 2, various types of carriers such as flies, ticks, mites, snails and so on, are responsible for the spread of infectious diseases in human populations who live in the region where densities of these carriers increase due to household discharges into the environment (Harold 1960, Harry and John 1962, Harry and Kent 1961). Some examples of such diseases are dysentery, gastroenteritis, diarrhea, cholera, measles, tuberculosis (Cairnoss and Feachem 1983, Taylor and Knowelden 1964). The migration of populations from environmentally degraded regions to cleaner regions also plays a very important role in the spread of infectious diseases as infected persons act as carriers or reservoirs of infection.

Migration of the population is a common phenomenon in the same region, country or even outside the country due to considerations which may be economic, social, religious, political, environmental and so on. The migrating population carries with it all of its

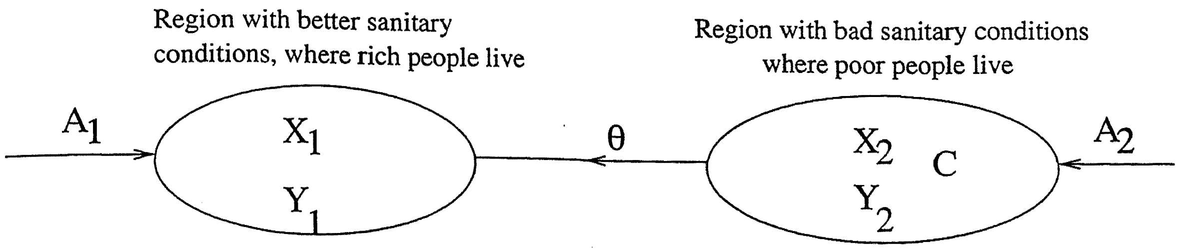


Figure 6.1: Spread of disease due to immigration.

traditional values and cultural heritage, including any diseases that are present in the population. After movement of the population into the new habitat, the susceptibles join the new susceptible class and the infectives join the new infective class and the usual spread of infection begins.

The effect of migration of the population on the spread of infectious diseases through demographic changes has been studied in some cases (Bailey 1980). But the effect of migration of the population from an environmentally degraded region or habitat to an environmentally cleaner region has not been modeled and analyzed so far.

In this chapter, therefore, we propose and analyze a model for the spread of an infectious disease between two socially structured populations (rich and poor). The rich population lives in a region with better sanitary conditions whilst the poor population lives in a region with bad sanitary conditions which is further affected by various household discharges conducive to the growth of a carrier population. The effect of a carrier on the spread of the disease in the region is also included in the proposed model as discussed in Chapter 2.

6.2 Mathematical Model

We consider that the population density N_1 of the rich class is divided into two subclasses of susceptibles X_1 and infectives Y_1, who live in a region with better sanitary conditions of the habitat or city. The population of poor people with density N_2 having susceptibles X_2 and infectives Y_2 live in a degraded region of the habitat with bad sanitary conditions

which is also affected by various household discharges. In the degraded region the effect of the carrier population in the model is considered along the same lines as described in Chapter 2. Thus a mathematical model is proposed as follows (Fig. 6.1),

$$\begin{aligned}
\dot{X}_1 &= A_1 - d_1X_1 - \beta_1X_1Y_1 + \nu_1Y_1 + \theta X_2, \\
\dot{Y}_1 &= \beta_1X_1Y_1 - (\nu_1 + \alpha_1 + d_1)Y_1 + \theta Y_2, \\
\dot{N}_1 &= A_1 - d_1N_1 - \alpha_1Y_1 + \theta N_2, \\
\dot{X}_2 &= A_2 - d_2X_2 - (\beta_2Y_2 + \lambda_2C)X_2 - \theta X_2 + \nu_2Y_2, \\
\dot{Y}_2 &= (\beta_2Y_2 + \lambda_2C)X_2 - (\nu_2 + \alpha_2 + d_2)Y_2 - \theta Y_2, \\
\dot{N}_2 &= A_2 - d_2N_2 - \alpha_2Y_2 - \theta N_2, \\
\dot{C} &= sC\left(1 - \frac{C}{L}\right) - s_0C + s_1EC, \\
\dot{E} &= Q(N_2) - \delta_0E,
\end{aligned} \tag{6.1}$$

where $s > s_0, \quad \theta > 0,$

$X_1(0) = X_{10} > 0, \quad Y_1(0) = Y_{10} \geq 0, \quad X_2(0) = X_{20} > 0, \quad Y_2(0) = Y_{20} \geq 0,$

$C(0) = C_0 > 0, \quad E(0) = E_0 > 0.$

Here, C is the density of the carrier population and E is the cumulative concentration of household discharges which are conducive to the growth of the carrier population; A_1 and A_2 are the constant rates of recruitment/immigration into the respective populations of susceptible class; d_1 and d_2 are the natural death rates of the richer and the poorer classes respectively; α_1 and α_2 are the disease related death rates of the richer and the poorer classes respectively; ν_1 and ν_2 are the recovery rates of richer and poor populations; θ is the constant rate of migration of people from the environmentally degraded region to the cleaner region of the habitat; β_1 and β_2 are the rates of contacts of infectives with susceptibles; λ_2 is the contact rate of poor class susceptible with the carriers. As before, the cumulative rate of environmental discharges $Q(N_2)$ is such that $\frac{dQ}{dN_2} > 0$. We take $Q(N_2) = Q_0 + lN_2$, where l is a constant.

In writing the model (6.1), we have assumed that the susceptibles and infectives from the degraded environment join the susceptibles and infectives of environmentally better habitat and the rate of immigration is proportional to the respective density of the

population.

We may note that the last four equations of the model (6.1) govern the growth of the infectious disease in the reservoir poor population (as discussed in Chapter 2) affecting the rich population in the habitat. Our main aim here is to study the effect of migration, particularly θ on the spread of the disease in the rich population.

We see that the region of attraction

$$\begin{aligned} T &= \left\{(Y_1, N_1, Y_2, N_2, C, E) : 0 \leq Y_1 \leq N_1 \leq \frac{(d_2+\theta)A_1 + \theta A_2}{d_1(d_2+\theta)}, \right. \\ &\qquad \left. 0 \leq Y_2 \leq N_2 \leq \frac{A_2}{d_2+\theta},\ 0 \leq C \leq C_{\max},\ 0 \leq E \leq E_{\max} \right\}, \end{aligned}$$

where $C_{\max} = \frac{L}{s}\left(s - s_0 + s_1 E_{\max}\right)$ and $E_{\max} = \frac{Q(\frac{A_2}{d_2+\theta})}{\delta_0}$, is positively invariant and all solutions starting in this region T stay in T. The continuity of the right hand sides of (6.1) and their derivatives imply that a unique solution exists (Hale 1969).

6.3 Case I: Q is a Constant Q_a

In this case, we consider that the rate of cumulative environmental discharges Q is independent of the population density N_2. Since $X_1 + Y_1 = N_1$ and $X_2 + Y_2 = N_2$, it is sufficient to consider the following subsystem of model (6.1) with the same initial conditions for its analysis.

$$\begin{aligned} \dot{Y}_1 &= \beta_1 Y_1 (N_1 - Y_1) - (\nu_1 + \alpha_1 + d_1) Y_1 + \theta Y_2, \\ \dot{N}_1 &= A_1 - d_1 N_1 - \alpha_1 Y_1 + \theta N_2, \\ \dot{Y}_2 &= (\beta_2 Y_2 + \lambda_2 C)(N_2 - Y_2) - (\nu_2 + \alpha_2 + d_2) Y_2 - \theta Y_2, \\ \dot{N}_2 &= A_2 - d_2 N_2 - \alpha_2 Y_2 - \theta N_2, \\ \dot{C} &= sC\left(1 - \frac{C}{L}\right) - s_0 C + s_1 E C, \\ \dot{E} &= Q_a - \delta_0 E. \end{aligned} \tag{6.2}$$

As discussed in Chapter 2, we use the asymptotic values of C and E in the rest of the equations of the system (6.2) and consider further the following subsystem with the same

initial conditions for our analysis.

$$\begin{aligned}
\dot{Y_1} &= \beta_1 Y_1(N_1 - Y_1) - (\nu_1 + \alpha_1 + d_1)Y_1 + \theta Y_2, \\
\dot{N_1} &= A_1 - d_1 N_1 - \alpha_1 Y_1 + \theta N_2, \\
\dot{Y_2} &= (\beta_2 Y_2 + \lambda_2 C_m)(N_2 - Y_2) - (\nu_2 + \alpha_2 + d_2)Y_2 - \theta Y_2, \\
\dot{N_2} &= A_2 - d_2 N_2 - \alpha_2 Y_2 - \theta N_2,
\end{aligned} \tag{6.3}$$

where $C_m = \frac{L}{s}\{s - s_0 + s_1 E_m\}$ and $E_m = \frac{Q_a}{\delta_0}$.

We note that the system (6.3) has only the nontrivial equilibrium, which is stated in the following theorem.

THEOREM 6.1 There exists an unique nontrivial (endemic) equilibrium point, namely, $P_1(\hat{Y_1}, \hat{N_1}, \hat{Y_2}, \hat{N_2})$, *where*

$$\hat{Y_1} = \frac{B_1 + \sqrt{B_1^2 + 4(d_2+\theta)(d_1+\alpha_1)\beta_1 d_1 (d_2+\theta)\theta \hat{Y_2}}}{2(d_2+\theta)(d_1+\alpha_1)\beta_1},$$

$$B_1 = [(d_2+\theta)\{\beta_1 A_1 - d_1(\nu_1 + \alpha_1 + d_1)\} + \theta A_2 \beta_1 - \beta_1 \theta \alpha_2 \hat{Y_2}],$$

$$\hat{Y_2} = \frac{B + \sqrt{B^2 + 4\beta_2(\alpha_2 + d_2 + \theta)\lambda_2 C_m A_2}}{2\beta_2(\alpha_2 + d_2 + \theta)},$$

$$B = \{\beta_2 A_2 - (d_2+\theta)(\lambda_2 C_m + \nu_2 + \alpha_2 + d_2 + \theta) - \alpha_2 \lambda_2 C_m\},$$

$$\hat{N_1} = \frac{(d_2+\theta)(A_1 - \alpha_1 \hat{Y_1}) + \theta(A_2 - \alpha_2 \hat{Y_2})}{d_1(d_2+\theta)} \quad \textit{and} \quad \hat{N_2} = \frac{(A_2 - \alpha_2 \hat{Y_2})}{(d_2+\theta)}.$$

Proof: Setting the right hand side of the equations in (6.3) to zero and with some manipulation, we get,

$$N_1 = \frac{(d_2+\theta)(A_1 - \alpha_1 Y_1) + \theta(A_2 - \alpha_2 Y_2)}{d_1(d_2+\theta)}, \tag{6.4}$$

$$N_2 = \frac{(A_2 - \alpha_2 Y_2)}{(d_2+\theta)}, \tag{6.5}$$

$$(d_2+\theta)(d_1+\alpha_1)\beta_1 Y_1^2 + [(d_2+\theta)\{d_1(\nu_1+\alpha_1+d_1) - \beta_1 A_1\} - \theta A_2 \beta_1 + \beta_1 \theta \alpha_2 Y_2]Y_1 - d_1(d_2+\theta)\theta Y_2 = 0 \tag{6.6}$$

and $\beta_2(\alpha_2+d_2+\theta)Y_2^2+[(d_2+\theta)(\lambda_2C_m+\nu_2+\alpha_2+d_2+\theta)+\alpha_2\lambda_2C_m-\beta_2A_2]Y_2-\lambda_2C_mA_2=0,$
(6.7)

which gives only one positive root $\hat{Y_2}$ as follows,

$$\hat{Y_2}=\frac{B+\sqrt{B^2+4\beta_2(\alpha_2+d_2+\theta)\lambda_2C_mA_2}}{2\beta_2(\alpha_2+d_2+\theta)},$$

where $B=\{\beta_2A_2-(d_2+\theta)(\lambda_2C_m+\nu_2+\alpha_2+d_2+\theta)-\alpha_2\lambda_2C_m\}$.
Using this value of $\hat{Y_2}$ in equation (6.6), we get corresponding positive root $\hat{Y_1}$ as follows,

$$\hat{Y_1}=\frac{B_1+\sqrt{B_1^2+4\beta_1d_1\theta(d_2+\theta)^2(d_1+\alpha_1)\hat{Y_2}}}{2\beta_1(d_2+\theta)(d_1+\alpha_1)},$$

where $B_1=[(d_2+\theta)\{\beta_1A_1-d_1(\nu_1+\alpha_1+d_1)\}+\theta A_2\beta_1-\beta_1\theta\alpha_2\hat{Y_2}]$.
Then $\hat{N_1}$ and $\hat{N_2}$ can be found by putting values of $\hat{Y_1}$ and $\hat{Y_2}$ in equations (6.4) and (6.5). For $\hat{N_1}$ and $\hat{N_2}$ to be positive, we should have conditions as $\hat{Y_1}<\frac{A_1}{\alpha_1}+\left(\frac{\theta}{d_2+\theta}\right)\left(\frac{A_2-\alpha_2\hat{Y_2}}{\alpha_1}\right)$ and $\hat{Y_2}<\frac{A_2}{\alpha_2}$, which are satisfied in view of (6.6) and (6.7). Thus we have only one equilibrium point namely $P_1(\hat{Y_1},\hat{N_1},\hat{Y_2},\hat{N_2})$.

Remark: We note from (6.7) that $\hat{Y_2}$ increases as C_m increases. Also from (6.6), we note that $\left(\frac{dY_1}{dY_2}\right)$ is positive only when $Y_1<\frac{d_1(d_2+\theta)}{\beta_1\alpha_2}$, which implies that an increase in $\hat{Y_2}$ causes an increase in $\hat{Y_1}$ provided that the above condition is satisfied (for α_2 very very small it seems to be satisfied).

6.3.1 Stability Analysis

Now we present the stability analysis of this equilibrium. The local stability result of this equilibrium is stated in the following theorem.

THEOREM 6.2 The equilibrium $P_1(\hat{Y_1},\hat{N_1},\hat{Y_2},\hat{N_2})$ is locally asymptotically stable.

Proof: In order to show the stability of P_1, we require the variational matrix M corresponding to the system (6.3) given by

$$M = \begin{pmatrix} m_{11} & \beta_1 Y_1 & \theta & 0 \\ -\alpha_1 & -d_1 & 0 & \theta \\ 0 & 0 & m_{33} & \beta_2 Y_2 + \lambda_2 C_m \\ 0 & 0 & -\alpha_2 & -(d_2+\theta) \end{pmatrix},$$

where $m_{11} = \beta_1 N_1 - (2\beta_1 Y_1 + \nu_1 + \alpha_1 + d_1)$ and
$m_{33} = \beta_2 N_2 - (2\beta_2 Y_2 + \lambda_2 C_m + \nu_2 + \alpha_2 + d_2 + \theta)$.
Now the variational matrix M_1 at the equilibrium point $P_1(\hat{Y}_1, \hat{N}_1, \hat{Y}_2, \hat{N}_2)$ is given by,

$$M_1 = \begin{pmatrix} -(\beta_1 \hat{Y}_1 + \theta \frac{\hat{Y}_2}{\hat{Y}_1}) & \beta_1 \hat{Y}_1 & \theta & 0 \\ -\alpha_1 & -d_1 & 0 & \theta \\ 0 & 0 & -\left(\beta_2 \hat{Y}_2 + \frac{\lambda_2 C_m \hat{N}_2}{\hat{Y}_2}\right) & \beta_2 \hat{Y}_2 + \lambda_2 C_m \\ 0 & 0 & -\alpha_2 & -(d_2+\theta) \end{pmatrix}.$$

The characteristic polynomial corresponding to the above matrix is given by

$$\left\{\psi^2 + (\beta_1 \hat{Y}_1 + \frac{\theta \hat{Y}_2}{\hat{Y}_1} + d_1)\psi + d_1(\beta_1 \hat{Y}_1 + \frac{\theta \hat{Y}_2}{\hat{Y}_1}) + \alpha_1 \beta_1 \hat{Y}_1\right\} \left\{\psi^2 + (\beta_2 \hat{Y}_2 + \lambda_2 C_m \frac{\hat{N}_2}{\hat{Y}_2}\right.$$
$$\left. + d_2 + \theta)\psi + (\beta_2 \hat{Y}_2 + \lambda_2 C_m \frac{\hat{N}_2}{\hat{Y}_2})(d_2+\theta) + \alpha_2(\beta_2 \hat{Y}_2 + \lambda_2 C_m)\right\} = 0.$$

Since the coefficients in both the quadratics are positive, the local stability of this equilibrium point is guaranteed by using the Routh-Hurwitz criteria.

Nonlinear Analysis and Simulation:

We first analyze the model (6.3) for $\alpha_1 = \alpha_2 = 0$, i.e. the disease related deaths in both the populations are zero. In this case, the model (6.3) with the same initial condition reduces to the following.

$$\begin{aligned} \dot{Y_1} &= \beta_1 Y_1 (N_1 - Y_1) - (\nu_1 + d_1) Y_1 + \theta Y_2, \\ \dot{N_1} &= A_1 - d_1 N_1 + \theta N_2, \\ \dot{Y_2} &= (\beta_2 Y_2 + \lambda_2 C_m)(N_2 - Y_2) - (\nu_2 + d_2) Y_2 - \theta Y_2, \\ \dot{N_2} &= A_2 - d_2 N_2 - \theta N_2, \end{aligned} \tag{6.8}$$

where $C_m = \frac{L}{s}\{s - s_0 + s_1 E_m\}$ and $E_m = \frac{Q_a}{\delta_0}$.
The equilibrium point corresponding to the above system (6.8) is given by $E^*((Y_1^*, N_1^*, Y_2^*, N_2^*)$,

where

$$N_2^* = \frac{A_2}{(d_2+\theta)}, \quad N_1^* = \frac{A_1+\theta N_2^*}{d_1},$$

$$Y_2^* = \frac{\{\beta_2 N_2^* - (\lambda_2 C_m + \nu_2 + d_2 + \theta)\} + \sqrt{\{\beta_2 N_2^* - (\lambda_2 C_m + \nu_2 + d_2 + \theta)\}^2 + 4\beta_2\lambda_2 C_m N_2^*}}{2\beta_2},$$

$$Y_1^* = \frac{\{\beta_1 N_1^* - (\nu_1 + d_1)\} + \sqrt{\{\beta_1 N_1^* - (\nu_1 + d_1)\}^2 + 4\beta_1\theta Y_2^*}}{2\beta_1}.$$

Here we can check that the equilibrium point $E^*(Y_1^*, N_1^*, Y_2^*, N_2^*)$ is globally stable by using the following Liapunov function.

$$V = \frac{1}{2}(Y_1 - Y_1^*)^2 + k_1\frac{1}{2}(N_1 - N_1^*)^2 + k_2\frac{1}{2}(Y_2 - Y_2^*)^2 + k_3\frac{1}{2}(N_2 - N_2^*)^2, \qquad (6.9)$$

where k_1, k_2 and k_3 are chosen as follows,

$$k_1 > \frac{A_1^2\beta_1^2 Y_1^*}{\theta Y_2^* d_1^3}, \quad k_2 > \frac{\theta Y_1^*}{\lambda_2 C_m N_2^*}, \qquad (6.10)$$

$$k_3 > \max\left\{\frac{k_1\theta}{d_1}, \frac{k_2 Y_2^*(\beta_2\frac{A_2}{d_2} + \lambda_2 C_m)^2}{(d_2+\theta)\lambda_2 C_m N_2^*}\right\}. \qquad (6.11)$$

We further note that the system (6.3) is bounded by the system (6.8), and hence using comparison theorems (Lakshmikantham and Leela, 1969), the solution of (6.3) will be bounded by the solution of (6.8). Hence we speculate that the nontrivial equilibrium point P_1 of the system (6.3) may be globally stable. Keeping this in view, we illustrate this result by simulation. The system (6.3) is integrated using the fourth order Runge-Kutta method and using the following set of parameters in the simulation (Greenhalgh 1990, 1992).

$\beta_1 = 0.00000051 = \beta_2$, $\nu_1 = 0.012$, $d_1 = 0.0004 = d_2$, $\theta = 0.0001$, $\alpha_1 = 0.0005$,

$A_1 = 10 = A_2$, $\lambda_2 = 0.000000021$, $\alpha_2 = 0.00052$, $\nu_2 = 0.011$, $C_m = 100000$.

The equilibrium values of $\hat{Y_1}$, $\hat{N_1}$, $\hat{Y_2}$, and $\hat{N_2}$ have been found as,

$\hat{Y_1} = 1816.779$, $\hat{N_1} = 26672.209$, $\hat{Y_2} = 4064.687$, $\hat{N_2} = 15772.727$.

Simulation is performed for different initial positions,

In A, $Y_{10} = 6981$, $N_{10} = 20377$, $Y_{20} = 7301$, $N_{20} = 17150$.

In B, $Y_{10} = 100$, $N_{10} = 10000$, $Y_{20} = 400$, $N_{20} = 21000$.

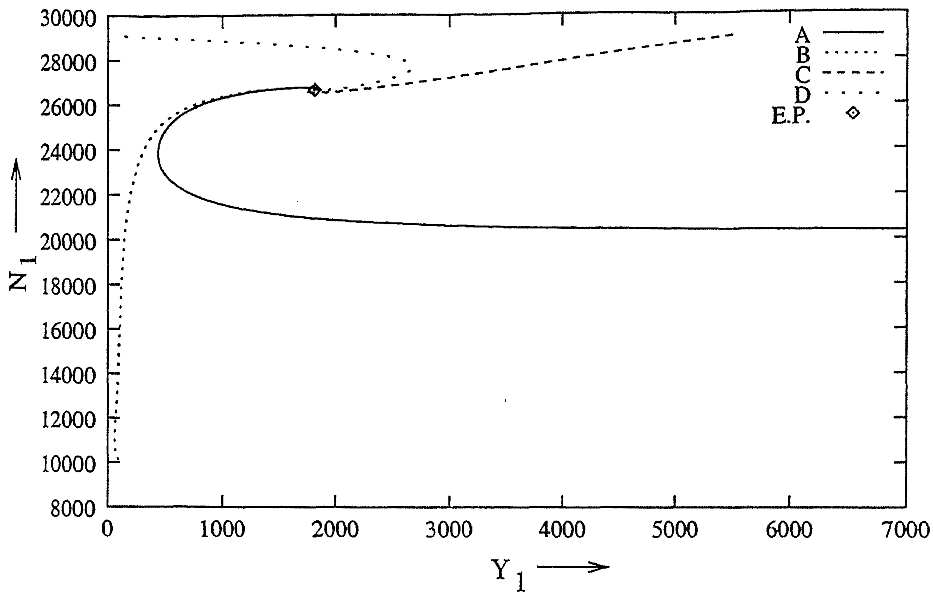


Figure 6.2: Variation of the total population with the infective population in the richer class.

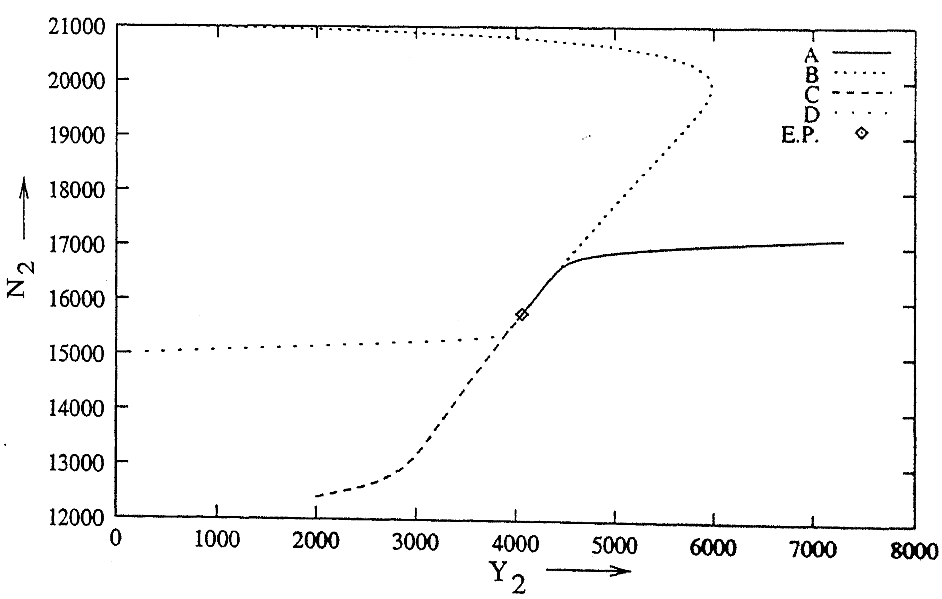


Figure 6.3: Variation of the total population with the infective population in the poorer class.

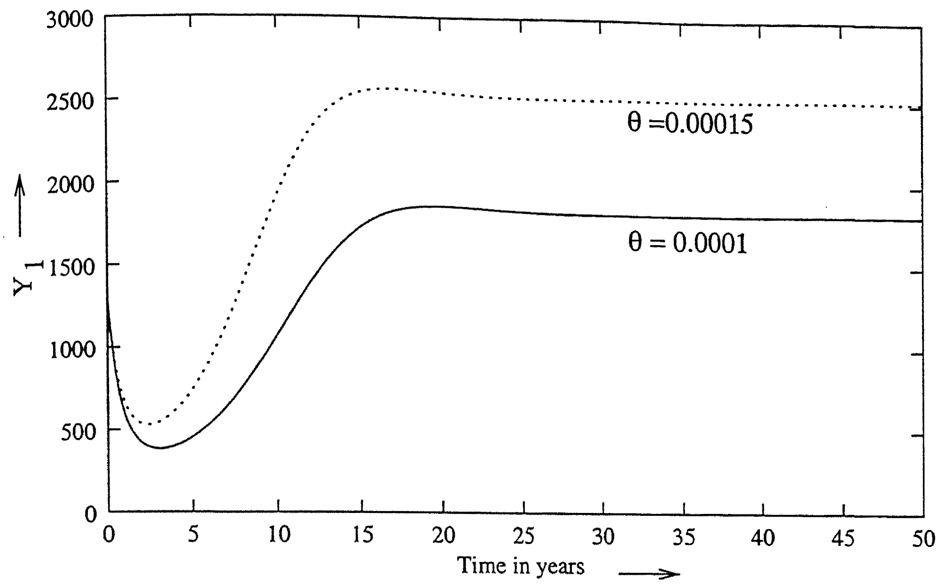


Figure 6.4: Variation of the infective population in the richer class with time.

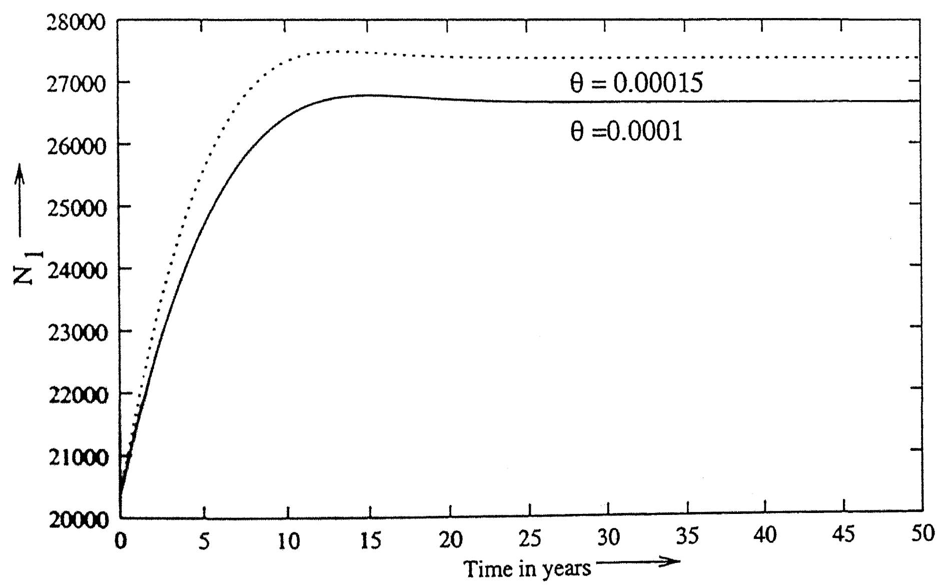


Figure 6.5: Variation of the total population in the richer class with time.

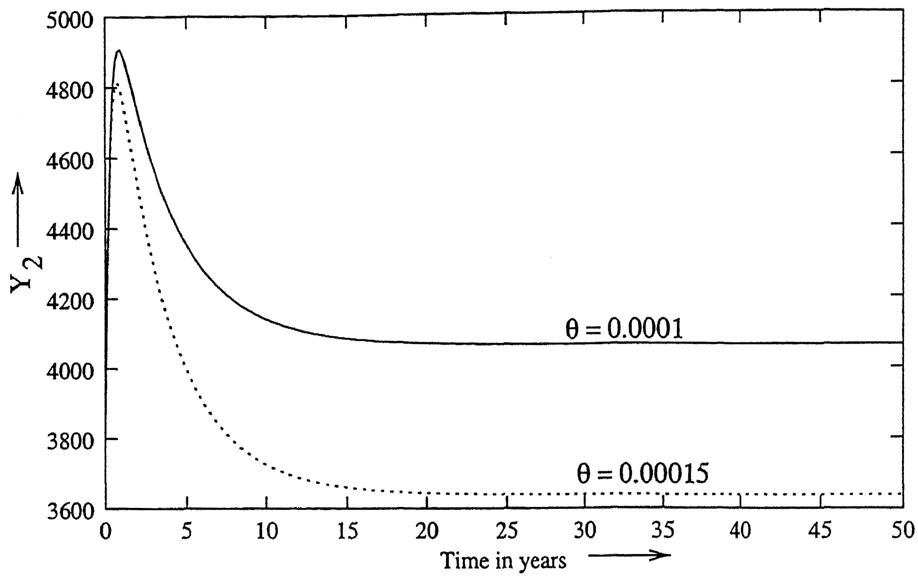


Figure 6.6: Variation of the infective class in the poorer class with time.

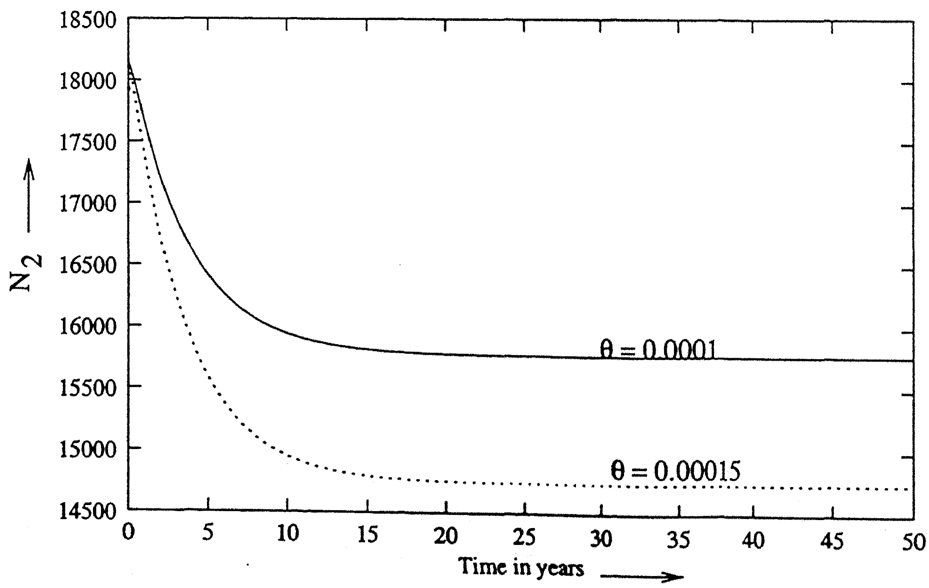


Figure 6.7: Variation of the total population in the poorer class with time.

In C, $Y_{10} = 5500, \quad N_{10} = 29000, \quad Y_{20} = 2000, \quad N_{20} = 12400.$

In D, $Y_{10} = 130, \quad N_{10} = 29100, \quad Y_{20} = 100, \quad N_{20} = 15000.$

In Figs. 6.2 and 6.3, we have plotted the total population against the infective population of the respective classes. From the solution curves, we conclude that the system appears to be globally stable about the endemic equilibrium point $(\hat{Y}_1, \ \hat{N}_1, \ \hat{Y}_2, \ \hat{N}_2)$ for this set of parameters. Also in Figs. 6.4-6.7 the effect of migration is shown on populations in the two regions. It is shown that as θ increases, the total population and the infective population of rich people increase and the total population and the infective population of poor people decrease.

Remark: Mena-Lorca and Hethcote (1992) determined a threshold σ for a single population without carrier as $\frac{\beta_1 \frac{A_1}{d_1}}{\nu_1+\alpha_1+d_1}$. The disease grows in the population only if $\sigma > 1$, otherwise if $\sigma < 1$ disease dies out and the disease free equilibrium is globally stable. We note that for above set of parameters, $\frac{\beta_1 \frac{A_1}{d_1}}{\nu_1+\alpha_1+d_1} < 1$. Here it is also observed that in absence of θ, i.e. $\theta = 0$, disease would die out in richer class. But for $\theta \neq 0$, disease persists in the richer class and regulates the population density.

6.4 Case II: Q is a Variable

We consider the case, when the rate of cumulative environmental discharges is dependent on the population density of the environmentally degraded region. Then in this case, it is sufficient to study the following subsystem of system (6.1) having the same initial conditions,

$$
\begin{aligned}
\dot{Y}_1 &= \beta_1 Y_1(N_1 - Y_1) - (\nu_1 + \alpha_1 + d_1)Y_1 + \theta Y_2, \\
\dot{N}_1 &= A_1 - d_1 N_1 - \alpha_1 Y_1 + \theta N_2, \\
\dot{Y}_2 &= (\beta_2 Y_2 + \lambda_2 C)(N_2 - Y_2) - (\nu_2 + \alpha_2 + d_2)Y_2 - \theta Y_2, \\
\dot{N}_2 &= A_2 - d_2 N_2 - \alpha_2 Y_2 - \theta N_2, \\
\dot{C} &= sC(1 - \frac{C}{L}) - s_0 C + s_1 EC, \\
\dot{E} &= Q_0 + l N_2 - \delta_0 E.
\end{aligned} \tag{6.12}
$$

The result of equilibrium analysis is stated in the following theorem.

THEOREM 6.3 There exist the following four equilibria,

(i) $E_1\left(0, \dfrac{A_1(d_2+\theta)+\theta A_2}{d_1(d_2+\theta)}, 0, \dfrac{A_2}{d_2+\theta}, 0, \dfrac{Q_0+l\frac{A_2}{d_2+\theta}}{\delta_0}\right)$,

(ii) $E_2\left(Y_1^*, N_1^*, 0, \frac{A_2}{d_2+\theta}, 0, \dfrac{Q_0+l\frac{A_2}{d_2+\theta}}{\delta_0}\right)$, *which exists if* $\beta_1(\theta A_2+(d_2+\theta)A_1) > d_1(d_2+\theta)(\nu_1+\alpha_1+d_1)$,

where $Y_1^* = \dfrac{\theta\beta_2 A_2+(d_2+\theta)\{\beta_1 A_1 - d_1(\nu_1+\alpha_1+d_1)\}}{\beta_1(d_2+\theta)(d_1+\alpha_1)}$, $N_1^* = \dfrac{\beta_1 Y_1^*+\nu_1+\alpha_1+d_1}{\beta_1}$,

(iii) $E_3\left(Y_1^{**}, N_1^{**}, Y_2^{**}, N_2^{**}, 0, \dfrac{Q_0+lN_2^{**}}{\delta_0}\right)$, *which exists if* $\dfrac{\beta_2 A_2}{d_2+\theta} > (\nu_2+\alpha_2+d_2+\theta)$,

where $Y_2^{**} = \dfrac{\beta_2 A_2-(d_2+\theta)(\nu_2+\alpha_2+d_2+\theta)}{\beta_2(d_2+\theta+\alpha_2)}$, $N_2^{**} = \dfrac{A_2-\alpha_2 Y_2^{**}}{d_2+\theta}$,

Y_1^{**} *is the positive root of the quadratic (6.21), when* $Y_2 = Y_2^{**} < \frac{A_2}{\alpha_2}$,

$N_1^{**} = \dfrac{(d_2+\theta)(A_1-\alpha_1 Y_1^{**})+\theta(A_2-\alpha_2 Y_2^{**})}{d_1(d_2+\theta)}$ *and* $E^{**} = \dfrac{Q_0+lN_2^{**}}{\delta_0}$,

(iv) $E_4(\hat{Y}_1, \hat{N}_1, \hat{Y}_2, \hat{N}_2, \hat{C}, \hat{E})$, *which exists if* $\beta_2 > \dfrac{\lambda_2 L}{s}\dfrac{s_1 l\alpha_2}{\delta_0(d_2+\theta)}$.

Proof: Setting the right hand side of the equations in (6.12) to zero and with some manipulation, we get, $E = \frac{Q_0+lN_2}{\delta_0}$, $C = 0$ *or* $C = \frac{L}{s}\{s - s_0 + s_1\frac{Q_0+lN_2}{\delta_0}\}$ and

$$N_1 = \frac{(d_2+\theta)(A_1-\alpha_1 Y_1)+\theta(A_2-\alpha_2 Y_2)}{d_1(d_2+\theta)}, \tag{6.13}$$

$$N_2 = \frac{(A_2-\alpha_2 Y_2)}{d_2+\theta}, \tag{6.14}$$

$$(d_2+\theta)(d_1+\alpha_1)\beta_1 Y_1^2 + [(d_2+\theta)\{d_1(\nu_1+\alpha_1+d_1)-\beta_1 A_1\} - \theta A_2\beta_1 + \beta_1\theta\alpha_2 Y_2]Y_1 \\ -d_1(d_2+\theta)\theta Y_2 = 0, \tag{6.15}$$

$$(\beta_2 Y_2 + \lambda_2 C)(A_2 - \alpha_2 Y_2) = (d_2+\theta)[\beta_2 Y_2^2 + \lambda_2 C Y_2 + (\nu_2+\alpha_2+d_2+\theta)Y_2]. \qquad (6.16)$$

A. When C=0:

It is obvious that the following three equilibria exist.

(i) When $C=0$ and $Y_2=0$, we have two equilibrium points namely,

$$E_1\left(0, \frac{A_1(d_2+\theta)+\theta A_2}{d_1(d_2+\theta)}, 0, \frac{A_2}{d_2+\theta}, 0, \frac{Q_0 + l\frac{A_2}{d_2+\theta}}{\delta_0}\right),\ E_2\left(Y_1^*, N_1^*, 0, \frac{A_2}{d_2+\theta}, 0, \frac{Q_0 + l\frac{A_2}{d_2+\theta}}{\delta_0}\right),$$

where $Y_1^* = \dfrac{\theta\beta_1 A_2 + (d_2+\theta)\{\beta_1 A_1 - d_1(\nu_1+\alpha_1+d_1)\}}{\beta_1(d_2+\theta)(d_1+\alpha_1)}$, $N_1^* = \dfrac{\beta_1 Y_1^* + \nu_1+\alpha_1+d_1}{\beta_1}$.

E_2 exists if $\beta_1(\theta A_2 + (d_2+\theta)A_1) > d_1(d_2+\theta)(\nu_1+\alpha_1+d_1)$.

(ii) When $C=0,\quad Y_2 = Y_2^{**}$,

we get the third equilibrium point as $E_3\left(Y_1^{**}, N_1^{**}, Y_2^{**}, N_2^{**}, 0, \dfrac{Q_0 + lN_2^{**}}{\delta_0}\right)$ under the following condition

$$\frac{\beta_2 A_2}{d_2+\theta} > (\nu_2+\alpha_2+d_2+\theta),$$

where $Y_2^{**} = \dfrac{\beta_2 A_2 - (d_2+\theta)(\nu_2+\alpha_2+d_2+\theta)}{\beta_2(d_2+\theta+\alpha_2)}$, $N_2^{**} = \dfrac{A_2 - \alpha_2 Y_2^{**}}{d_2+\theta}$.

Y_1^{**} is the positive root of the quadratic (6.15) and then N_1^{**} and E^{**} are as follows:

$$N_1^{**} = \frac{(d_2+\theta)(A_1 - \alpha_1 Y_1^{**}) + \theta(A_2 - \alpha_2 Y_2^{**})}{d_1(d_2+\theta)} \text{ and } E^{**} = \frac{Q_0 + lN_2^{**}}{\delta_0}.$$

B. When $C \neq 0$,

In this case, we have $C = \frac{L}{s}(F - GY_2)$, which is positive as $Y_2 < \dfrac{A_2}{\alpha_2}$ for positive N_2, (using (6.12)),

where $F = s - s_0 + s_1\dfrac{Q_0}{\delta_0} + \dfrac{s_1 l A_2}{\delta_0(d_2+\theta)}$ and $G = \dfrac{s_1 l \alpha_2}{\delta_0(d_2+\theta)}$.

In such a case equation (6.22) reduces to

$$(\alpha_2 + d_2 + \theta_2)\left(\beta_2 - \lambda_2 \frac{L}{s} G\right) Y_2^2 - [\beta_2 A_2 - (d_2 + \theta)(\nu_2 + \alpha_2 + d_2 + \theta) - \frac{\lambda_2 L}{s}(\alpha_2 + d_2 + \theta)F - \frac{\lambda_2 L}{s} G A_2\Big] Y_2 - \lambda_2 A_2 \frac{L}{s} F = 0.$$

We assume that $\beta_2 > \lambda_2 \frac{L}{s} G$. For $l = 0$ this equation should reduce to the previous case. Since the constant term in the above quadratic is negative, this implies existence of one and only one positive root, say $\hat{Y}_2$.

Under the condition, $\beta_2 > \lambda_2 \frac{LG}{s}$, from (6.15) we find the corresponding value of $\hat{Y}_1$ and from (6.13) and (6.14), $\hat{N}_1$ and $\hat{N}_2$ are calculated. Thus we get a nontrivial equilibrium point $E_4(\hat{Y}_1, \hat{N}_1, \hat{Y}_2, \hat{N}_2, \hat{C}, \hat{E})$ provided $\beta_2 > \frac{\lambda_2 L}{s} G$.
Hence we have four equilibria namely, E_1, E_2, E_3 and E_4, where E_2 exists if $\beta_1(\theta A_2 + (d_2 + \theta)A_1) > d_1(d_2 + \theta)(\nu_1 + \alpha_1 + d_1)$, E_3 exists if $\frac{\beta_2 A_2}{(d_2+\theta)(\nu_2+\alpha_2+d_2+\theta)} > 1$ and E_4 exists provided $\beta_2 > \lambda_2 \frac{LG}{s}$.

6.4.1 Stability Analysis

We now present the stability analysis of these equilibria in the following. The linear stability results of these equilibria are stated in the following theorem.

THEOREM 6.4 The equilibria E_1, $E_2$and E_3 are unstable and the equilibrium E_4 is stable if $a_3(a_1 a_2 - a_0 a_3) - a_1^2 > 0$, where a_1, $a_2$and a_3 are given explicitly in the proof of the theorem.

Proof: The variational matrix M corresponding to the system of equations (6.12) at $(Y_1, N_1, Y_2, N_2, C, E)$ is given by

$$M = \begin{pmatrix} m_{11} & \beta_1 Y_1 & \theta & 0 & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & \theta & 0 & 0 \\ 0 & 0 & m_{33} & \beta_2 Y_2 + \lambda_2 C & \lambda_2(N_2 - Y_2) & 0 \\ 0 & 0 & -\alpha_2 & -(d_2 + \theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & s - s_0 + s_1 E - \frac{2s}{L} C & s_1 C \\ 0 & 0 & 0 & l & 0 & -\delta_0 \end{pmatrix},$$

where $m_{11} = \beta_1(N_1 - 2Y_1) - (\nu_1 + \alpha_1 + d_1)$, $m_{33} = \beta_2(N_2 - 2Y_2) - \lambda_2 C - (\nu_2 + \alpha_2 + d_2 + \theta)$.

The variational matrix M_1 at equilibrium point $E_1\left(0, \dfrac{A_1(d_2+\theta)+\theta A_2}{d_1(d_2+\theta)}, 0, \dfrac{A_2}{d_2+\theta}, 0, \dfrac{Q_0 + l\frac{A_2}{d_2+\theta}}{\delta_0}\right)$ is given by

$$M_1 = \begin{pmatrix} \overline{m}_{11} & 0 & \theta & 0 & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & \theta & 0 & 0 \\ 0 & 0 & \overline{m}_{33} & 0 & \lambda_2\frac{A_2}{d_2+\theta} & 0 \\ 0 & 0 & -\alpha_2 & -(d_2+\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & \{s - s_0 + \frac{s_1}{\delta_0}(Q_0 + l\frac{A_2}{d_2+\theta})\} & 0 \\ 0 & 0 & 0 & l & 0 & -\delta_0 \end{pmatrix},$$

where

$\overline{m}_{11} = \beta_1\left\{\dfrac{A_1(d_2+\theta)+\theta A_2}{d_1(d_2+\theta)}\right\} - (\nu_1 + \alpha_1 + d_1)$ and $\overline{m}_{33} = \beta_2\frac{A_2}{d_2+\theta} - (\nu_2 + \alpha_2 + d_2 + \theta)$.

It is easy to see that the matrix M_1 has one positive eigenvalue given by $\{s - s_0 + \frac{s_1}{\delta_0}(Q_0 + l\frac{A_2}{d_2+\theta})\}$. Hence E_1 is unstable.

The variational matrix M_2 at the equilibrium point E_2 is given by

$$M_2 = \begin{pmatrix} -\beta_1 Y_1^* & \beta_1 Y_1^* & \theta & 0 & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & \theta & 0 & 0 \\ 0 & 0 & m_{33}^* & 0 & \lambda_2\frac{A_2}{d_2+\theta} & 0 \\ 0 & 0 & -\alpha_2 & -(d_2+\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & s - s_0 + \frac{s_1}{\delta_0}(Q_0 + l\frac{A_2}{d_2+\theta}) & 0 \\ 0 & 0 & 0 & l & 0 & -\delta_0 \end{pmatrix},$$

where $m_{33}^* = \beta_2\frac{A_2}{d_2+\theta} - (\nu_2 + \alpha_2 + d_2 + \theta)$. The above matrix has one positive eigenvalue as $\{s - s_0 + \frac{s_1}{\delta_0}(Q_0 + l\frac{A_2}{d_2+\theta})\}$, implying instability of the equilibrium E_2.

The variational matrix M_3 at equilibrium point E_3 is given by

$$M_3 = \begin{pmatrix} m_{11}^{**} & \beta_1 Y_1^{**} & \theta & 0 & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & \theta & 0 & 0 \\ 0 & 0 & -\beta_2 Y_2^{**} & \beta_2 Y_2^{**} & \lambda_2(N_2^{**} - Y_2^{**}) & 0 \\ 0 & 0 & -\alpha_2 & -(d_2+\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & s - s_0 + s_1 E^{**} & 0 \\ 0 & 0 & 0 & l & 0 & -\delta_0 \end{pmatrix},$$

where $m_{11}^{**} = -\left(\beta_1 Y_1^{**} + \frac{\theta Y_2^{**}}{Y_1^{**}}\right)$.

Here also one eigenvalue, which is $s - s_0 + s_1 E^{**}$, is positive, so the equilibrium E_3 is unstable.

The variational matrix M_4 at equilibrium point E_4 is given by

$$M_4 = \begin{pmatrix} \hat{m_{11}} & \beta_1\hat{Y_1} & \theta & 0 & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & \theta & 0 & 0 \\ 0 & 0 & \hat{m_{33}} & \beta_2\hat{Y_2}+\lambda_2\hat{C} & \lambda_2(\hat{N_2}-\hat{Y_2}) & 0 \\ 0 & 0 & -\alpha_2 & -(d_2+\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{s}{L}\hat{C} & s_1\hat{C} \\ 0 & 0 & 0 & l & 0 & -\delta_0 \end{pmatrix},$$

where $\hat{m_{11}} = -\left(\beta_1\hat{Y_1} + \theta\frac{\hat{Y_2}}{\hat{Y_1}}\right)$ and $\hat{m_{33}} = -\left(\beta_2\hat{Y_2} + \frac{\lambda_2\hat{C}\hat{N_2}}{\hat{Y_2}}\right)$.

Two characteristic roots of the matrix M_4 are given by

$$\psi^2 + \left(\beta_1\hat{Y_1} + \theta\frac{\hat{Y_2}}{\hat{Y_1}} + d_1\right)\psi + \left(\beta_1\hat{Y_1} + \theta\frac{\hat{Y_2}}{\hat{Y_1}}\right)d_1 + \alpha_1\beta_1\hat{Y_1} = 0,$$

and the other four characteristic roots are given by

$$\psi^4 + a_3\psi^3 + a_2\psi^2 + a_1\psi + a_0 = 0,$$

where

$$\begin{aligned}
a_3 &= \beta_2\hat{Y_2} + \lambda_2\frac{\hat{C}\hat{N_2}}{\hat{Y_2}} + d_2 + \theta + \frac{s}{L}\hat{C} + \delta_0, \\
a_2 &= \left(\beta_2\hat{Y_2} + \lambda_2\frac{\hat{C}\hat{N_2}}{\hat{Y_2}}\right)(d_2 + \theta + \frac{s}{L}\hat{C} + \delta_0) + (d_2+\theta)\left(\frac{s}{L}\hat{C} + \delta_0\right) \\
&\quad + \frac{s}{L}\hat{C}\delta_0 + \alpha_2(\beta_2\hat{Y_2} + \lambda_2\hat{C}), \\
a_1 &= \left(\beta_2\hat{Y_2} + \lambda_2\frac{\hat{C}\hat{N_2}}{\hat{Y_2}}\right)(d_2+\theta)\frac{s}{L}\hat{C} + (d_2+\theta)\frac{s}{L}\hat{C}\delta_0 + \frac{s}{L}\hat{C}\delta_0\left(\beta_2\hat{Y_2} + \lambda_2\frac{\hat{C}\hat{N_2}}{\hat{Y_2}}\right) \\
&\quad + \delta_0\left(\beta_2\hat{Y_2} + \lambda_2\frac{\hat{C}\hat{N_2}}{\hat{Y_2}}\right)(d_2+\theta) + \alpha_2(\beta_2\hat{Y_2} + \lambda_2\hat{C})\left(\frac{s}{L}\hat{C} + \delta_0\right), \\
a_0 &= \left(\beta_2\hat{Y_2} + \lambda_2\frac{\hat{C}\hat{N_2}}{\hat{Y_2}}\right)(d_2+\theta)\frac{s}{L}\hat{C}\delta_0 + \alpha_2(\beta_2\hat{Y_2} + \lambda_2\hat{C})\frac{s}{L}\hat{C}\delta_0 + \alpha_2\lambda_2(\hat{N_2} - \hat{Y_2})s_1\hat{C}l > 0.
\end{aligned}$$

Here $a_3 > 0$ and $\begin{vmatrix} a_3 & a_1 \\ 1 & a_2 \end{vmatrix} > 0$, thus using the Routh-Hurwitz criteria we observe that the equilibrium point E_4 is locally asymptotically stable if the following conditions are satisfied,

$$a_3 > 0, \quad \begin{vmatrix} a_3 & a_1 \\ 1 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_3 & a_1 & 0 \\ 1 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0, \quad \begin{vmatrix} a_3 & a_1 & 0 & 0 \\ 1 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & 1 & a_2 & a_0 \end{vmatrix} > 0.$$

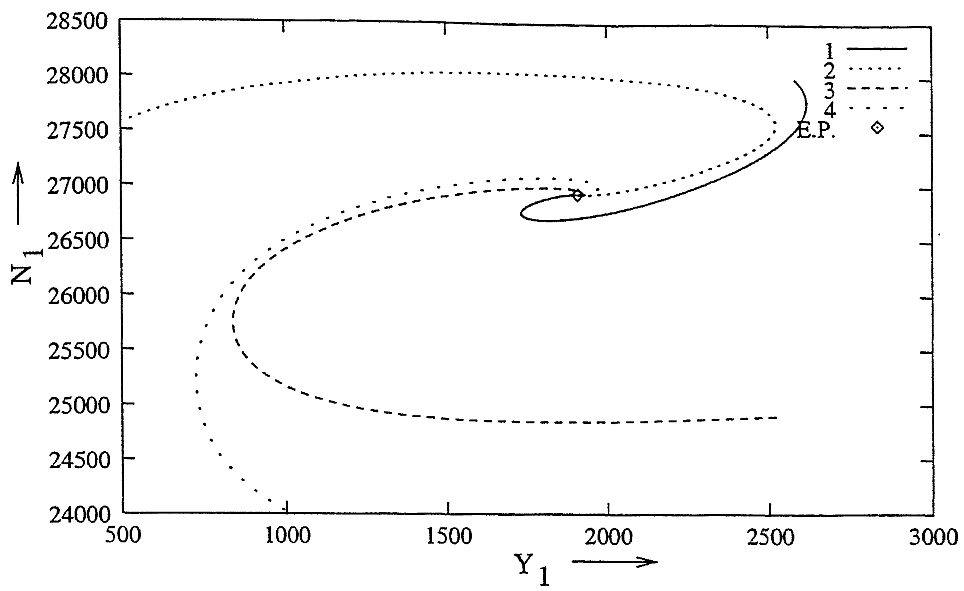


Figure 6.8: Variation of the total population with the infective population in the richer class.

We see that the first two inequalities are obvious. Also if the third inequality is satisfied so is the fourth one as $a_0 > 0$. Hence the equilibrium E_4 is locally asymptotically stable under the conditions stated in the theorem.

Nonlinear Analysis and Simulation:

As before we speculate that the system (6.12) is globally stable about the equilibrium E_4 under local stability conditions in the interior of the region of attraction. To illustrate this, the system (6.16) is integrated using the fourth order Runge-Kutta method using the following set of parameters in the simulation (Greenhalgh 1990, 1992), which satisfy local stability conditions.

$\beta_1 = 0.00000051 = \beta_2,\ \nu_1 = 0.012,\ d_1 = 0.0004 = d_2,\ \theta = 0.0001,\ \alpha_1 = 0.0005,$

$A_1 = 10 = A_2,\ \lambda_2 = 0.000000021,\ \alpha_2 = 0.00052,\ \nu_2 = 0.011, L = 100000,$

$s = 0.9,\ \ s_0 = 0.6,\ \ s_1 = 0.000002,\ Q_0 = 20, \delta_0 = 0.001,\ \ l = 0.00005.$

Here all parameters are in per day except the carrying capacity L and l.

Simulation is performed for different initial positions,

In 1, $Y_{10} = 2580,\ \ N_{10} = 28000,\ \ Y_{20} = 4000,\ \ N_{20} = 10600,\ \ C_0 = 8843,\ \ E_0 = 473.$

In 2, $Y_{10} = 520,\ \ N_{10} = 27590,\ \ Y_{20} = 3100,\ \ N_{20} = 23015,\ \ C_0 = 700,\ \ E_0 = 3000.$

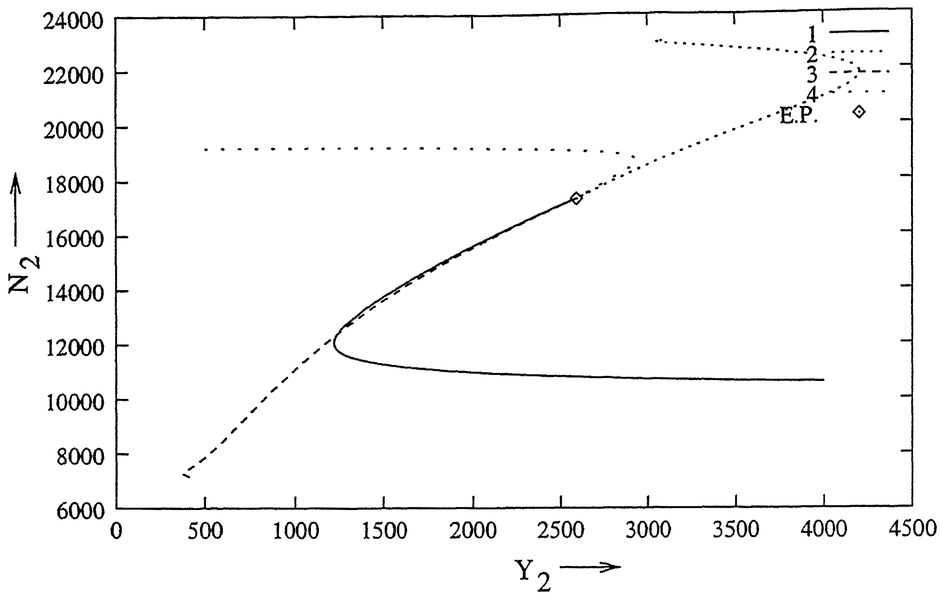


Figure 6.9: Variation of the total population with the infective population in the poorer class.

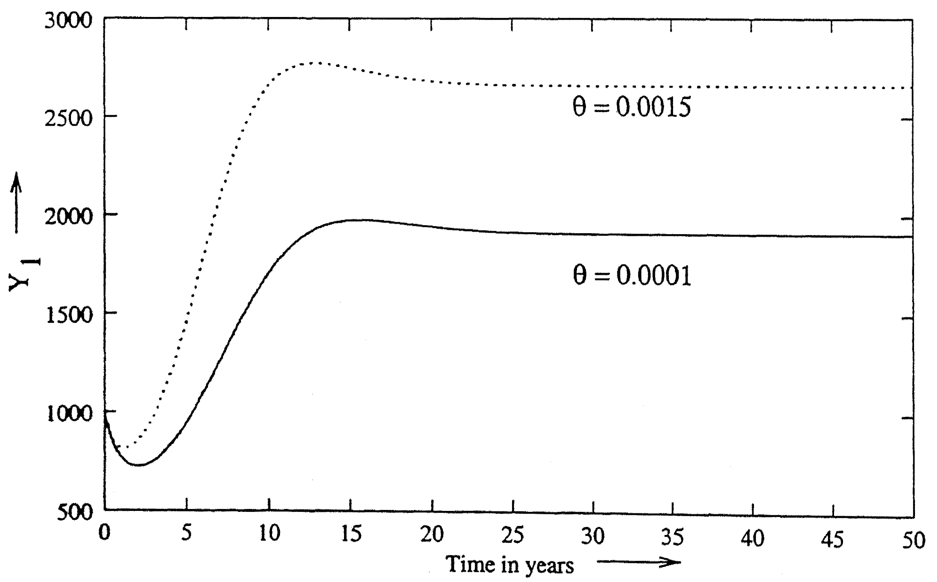


Figure 6.10: Variation of the infective population in the richer class with time.

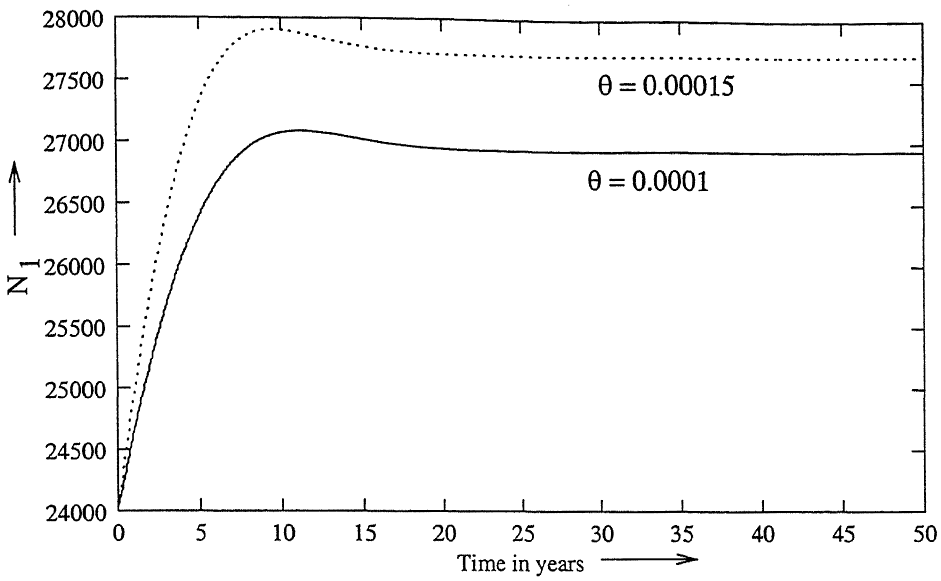


Figure 6.11: Variation of the total population in the richer class with time.

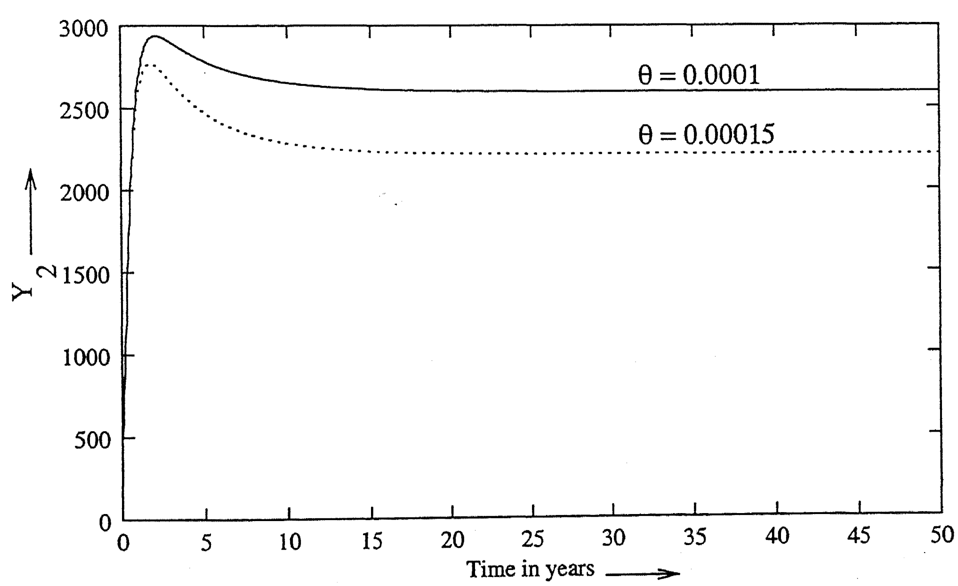


Figure 6.12: Variation of the infective population in the poorer class with time.

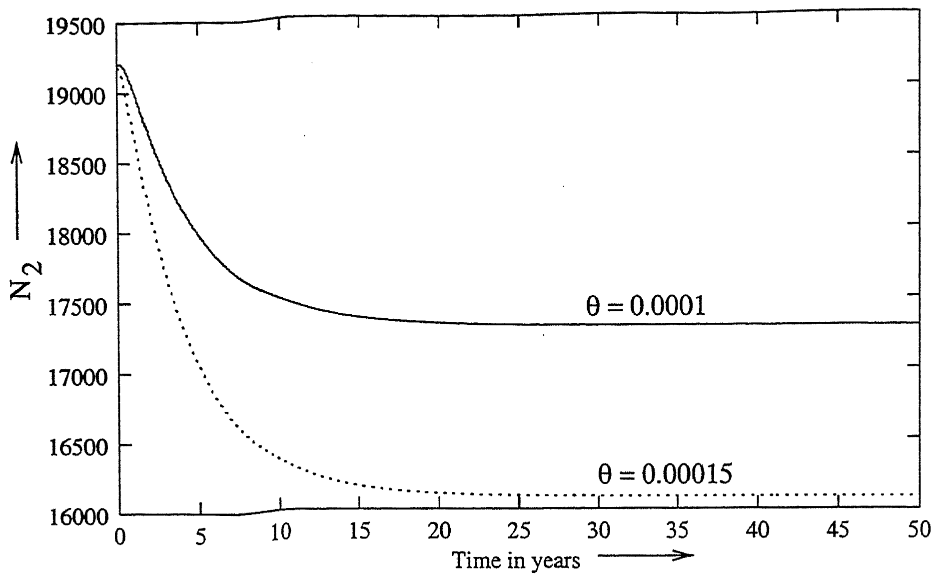


Figure 6.13: Variation of the total population in the poorer class with time.

In 3, $Y_{10} = 2520$, $N_{10} = 24902$, $Y_{20} = 410$, $N_{20} = 7155$, $C_0 = 1230$, $E_0 = 500$.
In 4, $Y_{10} = 1000$, $N_{10} = 24020$, $Y_{20} = 500$, $N_{20} = 19200$, $C_0 = 300$, $E_0 = 3400$.
The equilibrium values of $\hat{Y}_1$, $\hat{N}_1$, $\hat{Y}_2$, $\hat{N}_2$, $\hat{C}$ and $\hat{E}$ have been found as
$\hat{Y}_1 = 1910.05, \hat{N}_1 = 26937.81, \hat{Y}_2 = 2594.69, \hat{N}_2 = 17301.51, \hat{C} = 37970.01, \hat{E} = 20865.07$.
In Figs. 6.8 and 6.9, we have plotted the total population against the infective population of the respective classes. From the solution curves, we conclude that the system appears to be globally stable about the endemic equilibrium point ($\hat{Y}_1$, $\hat{N}_1$, $\hat{Y}_2$, $\hat{N}_2$, $\hat{C}$, $\hat{E}$) for the considered set of parameters, provided that we start away from other equilibria. In Figs. 6.10-6.13, the effects of θ on the infectives and the total population of the respective classes are shown. It is noted that as θ increases both the total population and the infectives in the environmentally better habitat increase while these decrease in environmentally degraded region, as expected.

Remark 1: The case when the migrating poor population works as service providers to the rich population and does not immigrate.

Consider the case when the migrating population living in a environmentally degraded region works as a service provider to the rich population living in a environmentally

cleaner region. Using the same notation as before, the model can be expressed as follows,

$$
\begin{aligned}
\dot{X}_1 &= A_1 - d_1 X_1 - \beta_1 X_1 Y_1 - \beta_{12} X_1 Y_2 + \nu_1 Y_1, \\
\dot{Y}_1 &= \beta_1 X_1 Y_1 + \beta_{12} X_1 Y_2 - (\nu_1 + \alpha_1 + d_1) Y_1, \\
\dot{X}_2 &= A_2 - d_2 X_2 - (\beta_2 Y_2 + \lambda_2 C) X_2 + \nu_2 Y_2, \\
\dot{Y}_2 &= (\beta_2 Y_2 + \lambda_2 C) X_2 - (\nu_2 + \alpha_2 + d_2) Y_2, \\
\dot{C} &= sC\left(1 - \frac{C}{L}\right) - s_0 C + s_1 E C, \\
\dot{E} &= Q(N_2) - \delta_0 E.
\end{aligned}
\tag{6.17}
$$

Since there is no immigration, θ is taken as zero and β_{12} is the rate of interaction with the infected poor population. In writing down the model it is assumed that the infected rich people do not interact with the service providers.

The analogy of the model (6.17) is similar to that discussed in Chapter 7, and hence it is not included in this chapter. The corresponding results can be visualized from the discussions and results presented in Chapter 7.

Remark 2: The same analysis can be done for the model with logistically growing human population. For a single population the logistic case has been done in Chapter 2, so one can easily extend this to the case of two populations as has been done in the case of constant immigration.

6.5 Conclusions

In this chapter, a non-linear mathematical model is proposed to study the effect of immigration of population, from an environmentally degraded habitat to the environmentally better habitat from sanitation point of view, on the spread of an infectious disease. This model is analyzed for two cases, (i) when the cumulative rate of environmental discharge is constant and (ii) when this rate is variable. In case I, we have shown that the non-trivial equilibrium point is locally asymptotically stable. Also by comparing the model of case I to the model when the disease related death rates are zero, we conjecture that this nontrivial equilibrium point is globally stable showing the endemic nature of the

disease in both the populations. The later result is also confirmed by computer simulation. In case II, the condition for existence of nontrivial equilibrium point has been obtained. It has been shown that this equilibrium point is locally asymptotically stable under certain conditions. By simulation it is shown that the nontrivial equilibrium is also globally asymptotically stable under local stability conditions for the set of parameters considered. In both the above cases, it is seen that as the migration rate θ increases, the total population and the infective class of the region of better environmental conditions increase, whereas the total population and the infective population of the region of degraded environmental condition decrease as expected. The analysis suggests that it is in the interest of rich population to clean the habitat of the poor population to avoid immigration on environmental consideration.

Chapter 7

Modelling the Spread of Bacterial Disease in a Population: Effect of Service Providers from an Environmentally Degraded Region

7.1 Introduction

Infectious diseases are the world's biggest killer of people and account for more than a dozen million deaths per year. The situation has become worse by uncontrolled increase in population due to a high birth rate as well as migration/movement of populations over the past several decades. Further the poor environmental conditions already existing in the densely populated cities of the third world countries have the greatest impact on the spread of bacterial disease such as tuberculosis, typhoid and so on. If the environment is conducive to the growth of the bacteria population, then this further helps in the spread of infectious diseases (Cairnross and Feachem 1983, Harold 1960, Taylor and Knowelden 1964).

In Chapter 6, we have modeled and analyzed the effect of immigration of a population from an environmentally degraded region on the spread of an infectious disease in a population living in a much cleaner environment. It may be noted here that when rich

and poor live in nearby habitats such as in big cities of the third world countries, then the poor people work as service providers in the houses of rich people but do not settle in the habitat of rich people. This population of poor people, as service providers, play an important role in the spread of infectious diseases as they carry pathogens in or on their bodies and may also transport disease vectors, such as lice. Organisms that survive primarily or entirely on a human host and are spread through sexual contact, droplet nuclei and close physical contact, can be readily carried to the susceptible population by these service providers, for example, AIDS, tuberculosis, measles, pertussis, diphtheria and hepatitis B (Morse 1995).

In this chapter, therefore, the spread of epidemics in a socially structured population (rich and poor) living in two nearby habitats, one cleaner than the other, is modeled and analyzed. It is assumed that the poor people suffer from an infectious disease (such as typhoid) caused by bacteria, the growth of which increases due to poor sanitation and various kinds of household discharges such as waste milk products, food, etc. It is assumed that the poor people living in the environmentally degraded habitat spread the disease by interacting with the rich population by working in their habitat as service providers. It may be noted here that the uncontrolled household discharges into the environment, not only increase the population of bacteria, but also affect the human population in several ways related to their general state of health, physical vitality, natural or acquired immunity and thereby making them more susceptible to various kinds of infectious diseases.

7.2 SIS Model with Immigration

We consider a general SIS model for infectious diseases caused by bacteria in a socially structured population (rich and poor) living in two adjoining habitats or neighborhoods. The environment where the rich people live is much cleaner, while the environment in the neighboring habitat where poor people live is not so clean and is very conducive to the growth of bacteria population caused by uncontrolled household discharges. It is

assumed further that infected poor people interact with rich people by working as various kinds of service providers and infect them. It is assumed further that the infected and sick rich people interact only with susceptibles of the rich class but not with people of the poor class. As in the previous chapter, here also the total population density of the rich class N_1 is divided into the susceptible class X_1 and the infective class Y_1. Similarly the total population density of the poor class N_2 is divided into the susceptible class X_2 and the infective class Y_2. It is assumed that all susceptibles X_2 of the poor population are affected by the bacteria whose density $B_2(t)$ grows logistically with given intrinsic growth rate and carrying capacity. It is considered that the growth of the bacteria population also increases due to increase in the cumulative density of environmental discharges, caused by sources independent or dependent on the human population in the environment. It is assumed that the bacteria population does not survive in the clean environment of rich people and only affects the population in the degraded environment of the poor class. In view of this, the model can be written as follows:

$$\begin{aligned}
\dot{X}_1 &= A_1 - d_1X_1 - \beta_1X_1Y_1 - \lambda_1X_1Y_2 + \nu_1Y_1,\\
\dot{Y}_1 &= \beta_1X_1Y_1 + \lambda_1X_1Y_2 - (\nu_1 + \alpha_1 + d_1)Y_1,\\
\dot{N}_1 &= A_1 - d_1N_1 - \alpha_1Y_1,\\
\dot{X}_2 &= A_2 - d_2X_2 - \beta_2X_2Y_2 - \lambda_2X_2B_2 + \nu_2Y_2,\\
\dot{Y}_2 &= \beta_2X_2Y_2 + \lambda_2X_2B_2 - (\nu_2 + \alpha_2 + d_2)Y_2,\\
\dot{N}_2 &= A_2 - d_2N_2 - \alpha_2Y_2, \qquad (7.1)\\
\dot{B}_2 &= sB_2\left(1 - \frac{B_2}{L}\right) + s_2Y_2 - s_{20}B_2 + s_3B_2E,\\
\dot{E} &= Q(N_2) - \delta_0E = Q_0 + lN_2 - \delta_0E,
\end{aligned}$$

$X_1(0) = X_{10} > 0,\ Y_1(0) = Y_{10} \geq 0,\ N_1(0) = N_{10} > 0,\ X_2(0) = X_{20} > 0,$
$Y_2(0) = Y_{20} \geq 0,\ N_2(0) > 0,\ B_2(0) \geq 0,\ E(0) = E_0 \geq 0.$

Here $E(t)$ is the cumulative density of household discharges conducive to the growth of the bacteria population; A_1 and A_2 are the immigration rate constants of the human population into the rich and poor populations respectively; d_1 and d_2 are the natural death rate constants corresponding to rich and poor populations; β_1 and λ_1 are the dis-

ease transmission coefficients in the rich population due to the infectives of the rich and poor populations respectively; β_2 and λ_2 are the disease transmission coefficients in the poor population due to the infectives in the poor population and bacteria respectively; α_1 and α_2 are the disease related death rate constants corresponding to rich and poor populations respectively; s is the intrinsic growth rate of the bacteria population; L is the carrying capacity of the bacteria population in the natural environment; s_{20} is the death rate of bacteria due to control measures, where $s > s_{20}$; s_2 is the rate of release of bacteria from the infective population; s_3 is the rate of growth of the bacteria population due to the environmental discharges; $Q(N_2)$ is the rate of cumulative environment discharges conducive to the growth of bacteria into the poor population which depends on the density N_2 of the poor population and δ_0 is the depletion rate coefficient of the cumulative environmental discharges. This model is analyzed for two types of environmental conditions:

(i) the rate of cumulative environmental discharges Q is a constant Q_a and

(ii) the rate of cumulative environmental discharges Q is a function of the poor population density N_2, assumed in the form $Q = Q_0 + lN_2$, l being a positive constant.

7.2.1 Case I: Q is a Constant Q_a

Since $X_1 + Y_1 = N_1$ and $X_2 + Y_2 = N_2$, the system (7.1) is equivalent to

$$
\begin{aligned}
\dot{Y_1} &= \beta_1(N_1 - Y_1)Y_1 + \lambda_1(N_1 - Y_1)Y_2 - (\nu_1 + \alpha_1 + d_1)Y_1,\\
\dot{N_1} &= A_1 - d_1N_1 - \alpha_1Y_1,\\
\dot{Y_2} &= \beta_2(N_2 - Y_2)Y_2 + \lambda_2(N_2 - Y_2)B_2 - (\nu_2 + \alpha_2 + d_2)Y_2,\\
\dot{N_2} &= A_2 - d_2N_2 - \alpha_2Y_2,\\
\dot{B_2} &= sB_2\left(1 - \frac{B_2}{L}\right) + s_2Y_2 - s_{20}B_2 + s_3B_2E,\\
\dot{E} &= Q_a - \delta_0E.
\end{aligned}
\tag{7.2}
$$

From the last equation of the above system of equations, we note that the asymptotic value of E is

$$\lim_{t\to\infty} E(t) = \frac{Q_a}{\delta_0}.$$

Therefore, in this case it is sufficient to study the following subsystem of (7.2) for its equilibrium analysis.

$$\begin{aligned}
\dot{Y_1} &= \beta_1(N_1 - Y_1)Y_1 + \lambda_1(N_1 - Y_1)Y_2 - (\nu_1 + \alpha_1 + d_1)Y_1,\\
\dot{N_1} &= A_1 - d_1 N_1 - \alpha_1 Y_1,\\
\dot{Y_2} &= \beta_2(N_2 - Y_2)Y_2 + \lambda_2(N_2 - Y_2)B_2 - (\nu_2 + \alpha_2 + d_2)Y_2,\\
\dot{N_2} &= A_2 - d_2 N_2 - \alpha_2 Y_2,\\
\dot{B_2} &= sB_2\left(1 - \frac{B_2}{L}\right) + s_2 Y_2 - s_{20}B_2 + s_3\frac{Q_a}{\delta_0}B_2.
\end{aligned} \tag{7.3}$$

The region of attraction T of the above system is

$$T = \left\{(Y_1, N_1, Y_2, N_2, B_2) : 0 \le Y_1 \le N_1 \le \frac{A_1}{d_1},\ 0 \le Y_2 \le \frac{A_2}{d_2},\ 0 \le B_2 \le B_{2\max}\right\},$$

where $B_{2\max} = \dfrac{L}{2s}\left[\left\{s - s_{20} + s_3\dfrac{Q_a}{\delta_0}\right\} + \sqrt{\left\{s - s_{20} + s_3\dfrac{Q_a}{\delta_0}\right\}^2 + \dfrac{4ss_2A_2}{d_2L}}\,\right]$,
is positively invariant and all solutions starting in this region T stay in T. The continuity of the right hand sides of (7.3) and their derivatives imply that a unique solution exists (Hale 1969).

The result of equilibrium analysis is stated in the following theorem.

THEOREM 7.1 There exist the following three equilibria of the model (7.3),

(i) $E_1(0, \frac{A_1}{d_1}, 0, \frac{A_2}{d_2}, 0)$,

(ii) $E_2(Y_1^*, N_1^*, 0, \frac{A_2}{d_2}, 0)$, *which exists if* $\frac{\beta_1 A_1}{d_1} > (\nu_1 + \alpha_1 + d_1)$,

where $Y_1^* = \dfrac{\beta_1\frac{A_1}{d_1} - (\nu_1 + \alpha_1 + d_1)}{\beta_1(1 + \frac{\alpha_1}{d_1})}$ *and* $N_1^* = \dfrac{A_1 - \alpha_1 Y_1^*}{d_1}$,

(iii) $E_3(\hat{Y}_1, \hat{N}_1, \hat{Y}_2, \hat{N}_2, \hat{B}_2)$.

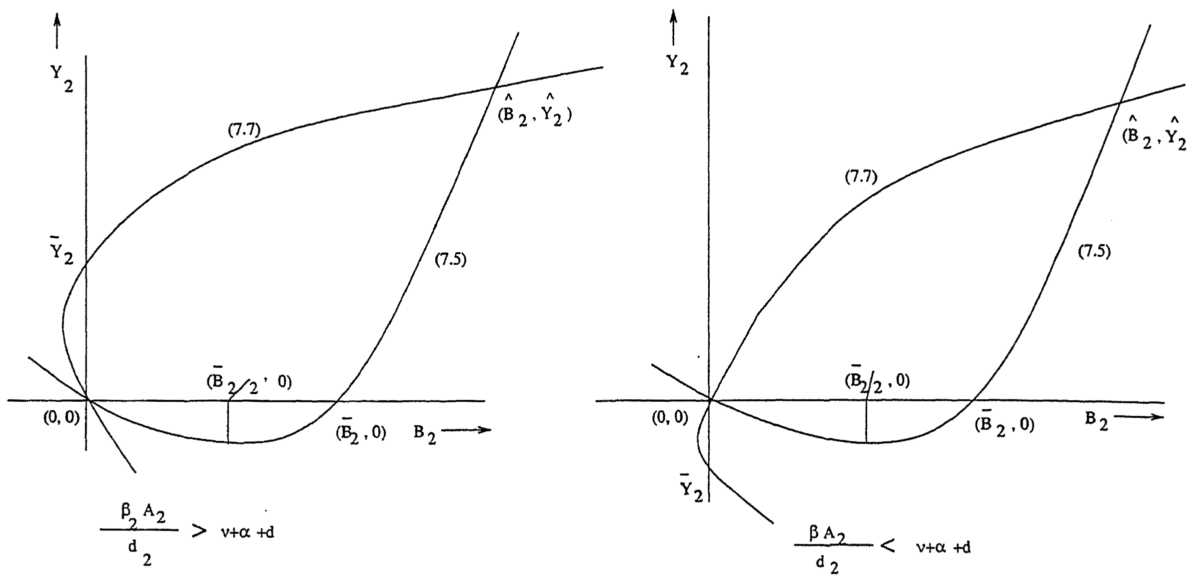


Figure 7.1: Existence of equilibrium point.

Proof : The existence of either of the first two equilibria is obvious. We prove the existence of the third equilibrium E_3 by the isocline method. Setting the right hand side of the equations in (7.3) to zero and with some manipulation, one gets:

$$N_1 = \frac{A_1 - \alpha_1 Y_1}{d_1}, \quad N_2 = \frac{A_2 - \alpha_2 Y_2}{d_2}, \tag{7.4}$$

$$Y_2 = \frac{1}{s_2}\left[\frac{s}{L}B_2^2 - \left(s - s_{20} + s_3\frac{Q_a}{\delta_0}\right)B_2\right], \tag{7.5}$$

$$\beta_1\left(1+\frac{\alpha_1}{d_1}\right)Y_1^2 - \left\{\beta_1\frac{A_1}{d_1} - (\nu_1+\alpha_1+d_1)\right\}Y_1 + \lambda_1\left(1+\frac{\alpha_1}{d_1}\right)Y_1Y_2 - \lambda_1\frac{A_1}{d_1}Y_2 = 0, \tag{7.6}$$

$$\beta_2\left(1+\frac{\alpha_2}{d_2}\right)Y_2^2 - \left\{\frac{\beta_2 A_2}{d_2} - (\nu_2+\alpha_2+d_2) - \lambda_2\left(1+\frac{\alpha_2}{d_2}\right)B_2\right\}Y_2 - \lambda_2\frac{A_2}{d_2}B_2 = 0. \tag{7.7}$$

Now we show the existence of $\hat{Y}_2$ and $\hat{B}_2$ from (7.5) and (7.7) and the corresponding values of $\hat{Y}_1, \hat{N}_1$ and $\hat{N}_2$ can be obtained from (7.4) and (7.6).

From equation (7.5): we have,

(i) $B_2 = 0$ or $B_2 = \frac{L}{s}\left\{s - s_{20} + s_3\frac{Q_a}{\delta_0}\right\} = \bar{B}_2 > 0$, when $Y_2 = 0$,

(ii) the slope of the curve (7.5) is given by

$$\frac{dY_2}{dB_2} = \frac{1}{s_2}\left\{\frac{2s}{L}B_2 - (s - s_{20} + s_3\frac{Q_a}{\delta_0})\right\}, \tag{7.8}$$

at $(0,0)$, the slope of (7.5) i.e. $\frac{dY_2}{dB_2}$ is negative and at $(\bar{B}_2, 0)$, it is positive and from (7.6), the slope of (7.5) is zero at $B_2 = \frac{L}{2s}(s - s_{20} + s_3\frac{Q_a}{\delta_0}) = \frac{\bar{B}_2}{2}$,
(iii) the slope $\frac{dY_2}{dB_2}$ is increasing as $\left(\frac{d^2Y_2}{dB_2^2}\right) > 0$.

From (7.7), the following points are observed.
(i) For $B_2 = 0$,

$$Y_2 = 0 \quad \text{or} \quad Y_2 = \frac{\beta_2\frac{A_2}{d_2} - (\nu_2 + \alpha_2 + d_2)}{\beta_2(1 + \frac{\alpha_2}{d_2})} = \bar{Y}_2 \text{ (say)},$$

which is positive if $\beta_2\frac{A_2}{d_2} > (\nu_2 + \alpha_2 + d_2)$ and negative otherwise.
(ii) At $(0,0)$, the slope of (7.7) is given by

$$\frac{dY_2}{dB_2} = -\frac{\lambda_2\frac{A_2}{d_2}}{\beta_2\frac{A_2}{d_2} - (\nu_2 + \alpha_2 + d_2)},$$

which is positive or negative depending upon $\bar{Y}_2$ being negative or positive respectively. At $(0, \bar{Y}_2)$, the slope is given by

$$\frac{dY_2}{dB_2} = \frac{\lambda_2\dfrac{\nu_2 + \alpha_2 + d_2}{\beta_2}}{\beta_2\dfrac{A_2}{d_2} - (\nu_2 + \alpha_2 + d_2)},$$

which is positive or negative depending upon $\bar{Y}_2$ being positive or negative respectively.
(iii) Slope $\frac{dY_2}{dB_2}$ is decreasing as $\left(\frac{d^2Y_2}{dB_2^2}\right) < 0$.
Thus, after plotting Y_2 with B_2 corresponding to (7.5) and (7.7) in Fig 7.1, we see that there are two intersecting points $(0,0)$ and $(\hat{B}_2, \hat{Y}_2)$. After finding $\hat{B}_2, \hat{Y}_2$ and $\hat{Y}_1$, we can calculate $\hat{N}_1$ and $\hat{N}_2$ using (7.6) and (7.4). Also, we note from (7.6) and (7.7) that $\hat{Y}_1 < \frac{A_1}{\alpha_1 + d_1}$ and $Y_2 < \frac{A_2}{\alpha_2 + d_2}$. Thus corresponding to $(\hat{B}_2, \hat{Y}_2)$, we have the third nontrivial equilibrium, namely $E_3(\hat{Y}_1, \hat{N}_1, \hat{Y}_2, \hat{N}_2, \hat{B}_2)$.

7.2.1.1 Stability Analysis

Now we present the stability analysis of these equilibria. The local stability results of these equilibria are stated in the following theorem.

THEOREM 7.2 The equilibria E_1 and E_2 are unstable and the equilibrium E_3 is locally asymptotically stable.

Proof : The variational matrix M at $(Y_1, N_1, Y_2, N_2, B_2)$ is given by

$$M = \begin{pmatrix} m_{11} & \beta_1 Y_1 + \lambda_1 Y_2 & \lambda_1(N_1 - Y_1) & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & 0 & 0 \\ 0 & 0 & m_{33} & \beta_2 Y_2 + \lambda_2 B_2 & \lambda_2(N_2 - Y_2) \\ 0 & 0 & -\alpha_2 & -d_2 & 0 \\ 0 & 0 & s_2 & 0 & m_{55} \end{pmatrix},$$

where

$m_{11} = \beta_1(N_1 - 2Y_1) - (\lambda_1 Y_2 + \nu_1 + \alpha_1 + d_1)$, $m_{33} = \beta_2(N_2 - 2Y_2) - (\lambda_2 B_2 + \nu_2 + \alpha_2 + d_2)$

and

$m_{55} = s - s_{20} + s_3 \frac{Q_a}{\delta_0} - \frac{2s}{L} B_2$.

The variational matrix M_1 at the equilibrium point E_1, is given by

$$M_1 = \begin{pmatrix} m_{11} & 0 & \lambda_1 \frac{A_1}{d_1} & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & 0 & 0 \\ 0 & 0 & m_{33} & 0 & \lambda_2 \frac{A_2}{d_2} \\ 0 & 0 & -\alpha_2 & -d_2 & 0 \\ 0 & 0 & s_2 & 0 & s - s_{20} + s_3 \frac{Q_a}{\delta_0} \end{pmatrix}.$$

It is easy to see that at least one eigenvalue of M_1 is positive or has positive real part. Hence the equilibrium E_1 is unstable.

The variational matrix M_2 at the equilibrium point E_2, is given by

$$M_2 = \begin{pmatrix} -\beta_1 Y_1^* & \beta_1 Y_1^* & \lambda_1(N_1^* - Y_1^*) & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & 0 & 0 \\ 0 & 0 & \beta_2 \frac{A_2}{d_2} - (\nu_2 + \alpha_2 + d_2) & 0 & \lambda_2 \frac{A_2}{d_2} \\ 0 & 0 & -\alpha_2 & -d_2 & 0 \\ 0 & 0 & s_2 & 0 & s - s_{20} + s_3 \frac{Q_a}{\delta_0} \end{pmatrix}.$$

The characteristic polynomial corresponding to the above matrix is given by

$$(d_2 + \psi)\{\psi^2 + (d_1 + \beta_1 Y_1^*)\psi + (d_1 + \alpha_1)\beta_1 Y_1^*\} \left[\psi^2 + h_1 \psi + h_2\right] = 0,$$

where

$$\begin{aligned} h_1 &= -\left\{\frac{\beta_2 A_2}{d_2} - (\nu_2 + \alpha_2 + d_2) + s - s_{20} + s_3 \frac{Q_a}{\delta_0}\right\} \\ h_2 &= \left\{\frac{\beta_2 A_2}{d_2} - (\nu_2 + \alpha_2 + d_2)\right\}\left(s - s_{20} + s_3 \frac{Q_a}{\delta_0}\right) - \frac{\lambda_2 s_2 A_2}{d_2}. \end{aligned}$$

Using the Routh-Hurwitz criteria, this equilibrium is stable, provided

$$\left\{\frac{\beta_2 A_2}{d_2} - (\nu_2 + \alpha_2 + d_2) + s - s_{20} + s_3 \frac{Q_a}{\delta_0}\right\} < 0$$

$$\text{and}\quad \left\{\frac{\beta_2 A_2}{d_2} - (\nu_2+\alpha_2+d_2)\right\}\left(s - s_{20} + s_3\frac{Q_a}{\delta_0}\right) - \frac{\lambda_2 s_2 A_2}{d_2} > 0.$$

We note that both the above conditions are simultaneously not satisfied, so this equilibrium is unstable.

The variational matrix M_3 at the equilibrium point E_3, is given by

$$M_3 = \begin{pmatrix} \hat{m_{11}} & \beta_1\hat{Y_1}+\lambda_1\hat{Y_2} & \lambda_1(\hat{N_1}-\hat{Y_1}) & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & 0 & 0 \\ 0 & 0 & \hat{m_{33}} & \beta_2\hat{Y_2}+\lambda_2\hat{B_2} & \lambda_2(\hat{N_2}-\hat{Y_2}) \\ 0 & 0 & -\alpha_2 & -d_2 & 0 \\ 0 & 0 & s_2 & 0 & \hat{m_{55}} \end{pmatrix},$$

where $\hat{m_{11}} = -\left(\beta_1\hat{Y_1} + \frac{\lambda_1\hat{N_1}\hat{Y_2}}{\hat{Y_1}}\right)$, $\hat{m_{33}} = -\left(\beta_2\hat{Y_2} + \frac{\lambda_2\hat{N_2}\hat{B_2}}{\hat{Y_2}}\right)$ and $\hat{m_{55}} = -\left(\frac{s}{L}\hat{B_2} + s_2\frac{\hat{Y_2}}{\hat{B_2}}\right)$.

The characteristic polynomial corresponding to the above matrix is given by

$$\left\{\psi^2 + \left(d_1+\beta_1\hat{Y_1}+\frac{\lambda_1\hat{N_1}\hat{Y_2}}{\hat{Y_1}}\right)\psi + d_1\left(\beta_1\hat{Y_1}+\frac{\lambda_1\hat{N_1}\hat{Y_2}}{\hat{Y_1}}\right)\right\}\left[\psi^3+a_1\psi^2+a_2\psi+a_3\right]=0, \tag{7.9}$$

where

$$\begin{aligned} a_1 &= d_2+\beta_2\hat{Y_2}+\frac{\lambda_2\hat{N_2}\hat{B_2}}{\hat{Y_2}}+\frac{s}{L}\hat{B_2}+s_2\frac{\hat{Y_2}}{\hat{B_2}} \;>\; 0, \\ a_2 &= d_2\left(\beta_2\hat{Y_2}+\frac{\lambda_2\hat{N_2}\hat{B_2}}{\hat{Y_2}}+\frac{s}{L}\hat{B_2}+s_2\frac{\hat{Y_2}}{\hat{B_2}}\right)+\left(\beta_2\hat{Y_2}+\frac{\lambda_2\hat{N_2}\hat{B_2}}{\hat{Y_2}}\right)\left(\frac{s}{L}\hat{B_2}+s_2\frac{\hat{Y_2}}{\hat{B_2}}\right) \\ &\quad +\alpha_2(\beta_2\hat{Y_2}+\lambda_2\hat{B_2})-\lambda_2(\hat{N_2}-\hat{Y_2})s_2 \;>\; 0, \\ a_3 &= d_2\left(\beta_2\hat{Y_2}+\frac{\lambda_2\hat{N_2}\hat{B_2}}{\hat{Y_2}}\right)\left(\frac{s}{L}\hat{B_2}+s_2\frac{\hat{Y_2}}{\hat{B_2}}\right)+\alpha_2(\beta_2\hat{Y_2}+\lambda_2\hat{B_2})\left(\frac{s}{L}\hat{B_2}+s_2\frac{\hat{Y_2}}{\hat{B_2}}\right) \\ &\quad -\lambda_2(\hat{N_2}-\hat{Y_2})d_2 s_2 \;>\; 0. \end{aligned}$$

In equation (7.9), the coefficients of the quadratic are positive which implies that the roots cannot have positive real parts. Also it can be checked that $a_1a_2 - a_3 > 0$, hence by the Routh-Hurwitz criteria, the roots of the cubic in equation (7.9) have negative real parts. Hence the system is locally asymptotically stable about this equilibrium E_3.

Nonlinear Analysis and Simulation: Before we proceed for simulation, it can be shown heuristically that the system (7.3) is globally stable about its nontrivial equilibrium, when $\alpha_1 = 0$ and $\alpha_2 = 0$ (see Appendix III). Since the system (7.3) is bounded by its corresponding system with $\alpha_1 = 0$ and $\alpha_2 = 0$, using a comparison theorem (Lakshmikantham and Leela 1969), it is concluded that the solution of the system (7.3) is

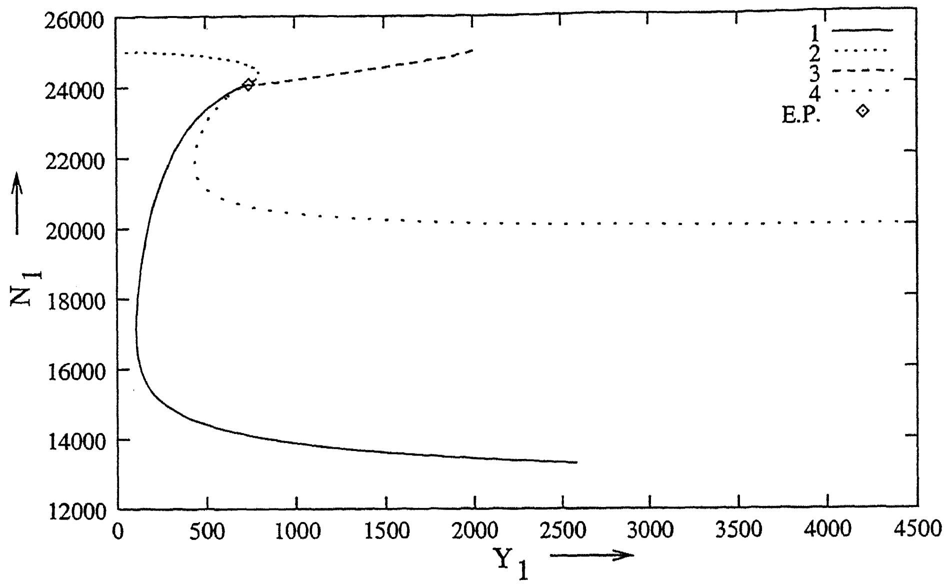


Figure 7.2: Variation of N_1 with Y_1.

bounded by the solution of (7.3) with $\alpha_i = 0,\ i = 1, 2$. So we speculate that the equilibrium E_3 of (7.3) may be globally stable, provided we start away from other equilibria. To illustrate this and to see the effects of various parameters on $\hat{Y_1}$, the system (7.3) is integrated using the fourth order Runge-Kutta Method and using the following parameter values in the simulation (Greenhalgh 1990, 1992),

$\beta_1 = 0.00000051,\quad \lambda_1 = 0.000000011,\quad \nu_1 = 0.012,\quad d_1 = 0.0004,$

$\alpha_1 = 0.0005,\quad A_1 = 10,\quad \beta_2 = 0.00000051,\quad \lambda_2 = 0.0000000002,$

$\alpha_2 = 0.00052,\quad \nu_2 = 0.011,\quad d_2 = 0.0004,\quad A_2 = 10,\quad s = 1,\quad s_{20} = 0.65,$

$s_2 = 10,\quad s_3 = 0.000002,\quad L = 5000000,\quad Q_a = 20,\quad \delta_0 = 0.001.$

The equilibrium values of $\hat{Y_1}$, $\hat{N_1}$, $\hat{Y_2}$, $\hat{N_2}$ and $\hat{B_2}$ are found as

$\hat{Y_1} = 741.710,\quad \hat{N_1} = 24072.893,\quad \hat{Y_2} = 2893.195,\quad \hat{N_2} = 21238.835,\quad \hat{B_2} = 2021558.647.$

Here all parameters are in per day except L and l which are constants. Simulation is performed for different initial positions 1, 2, 3 and 4 as shown in Figs. 7.2 and 7.3. In these figures, we have plotted the infective population against the total population of the respective classes. From the solution curves, we conclude that it is plausible that for the set of parameters considered and provided that we do not start at E_1 or E_2 the system tends to the endemic equilibrium ($\hat{Y_1}$, $\hat{N_1}$, $\hat{Y_2}$, $\hat{N_2}$, $\hat{B_2}$). The effects of various parameters on Y_1 and Y_2 are shown in Figs. 7.4-7.7. The following is concluded from

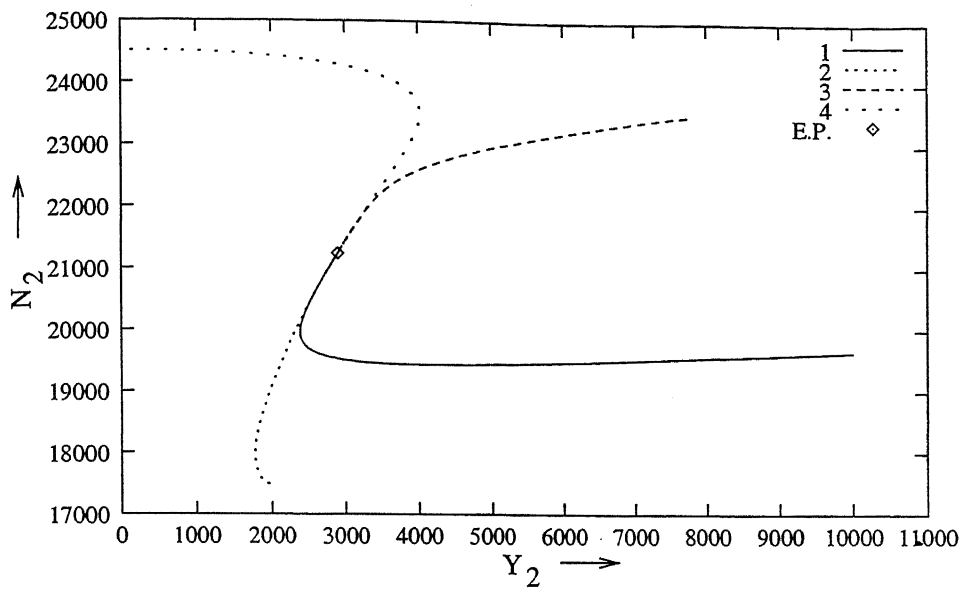


Figure 7.3: Variation of N_2 with Y_2.

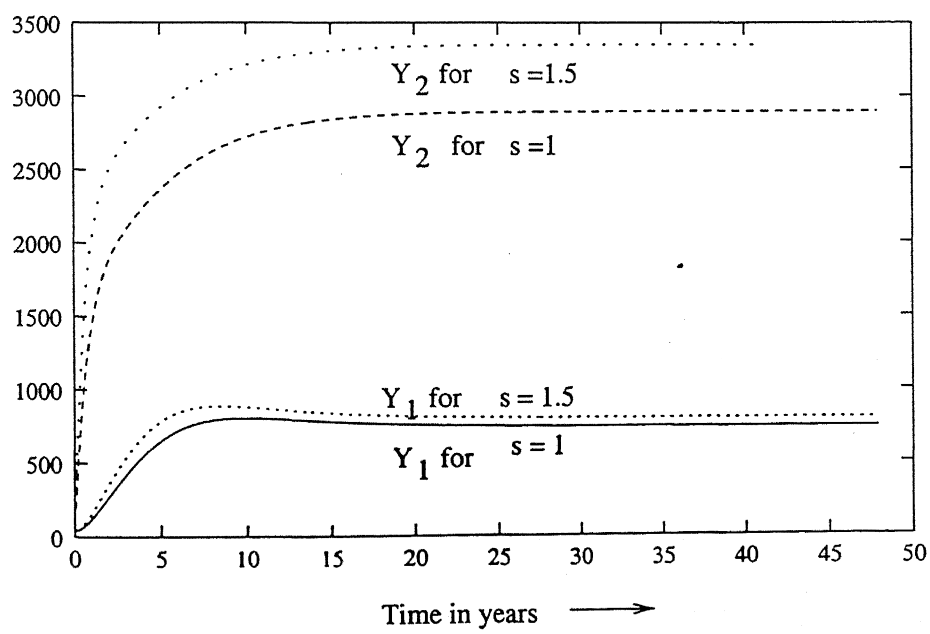


Figure 7.4: Variation of Y_1 and Y_2 with time for different intrinsic growth rates of the bacterial population.

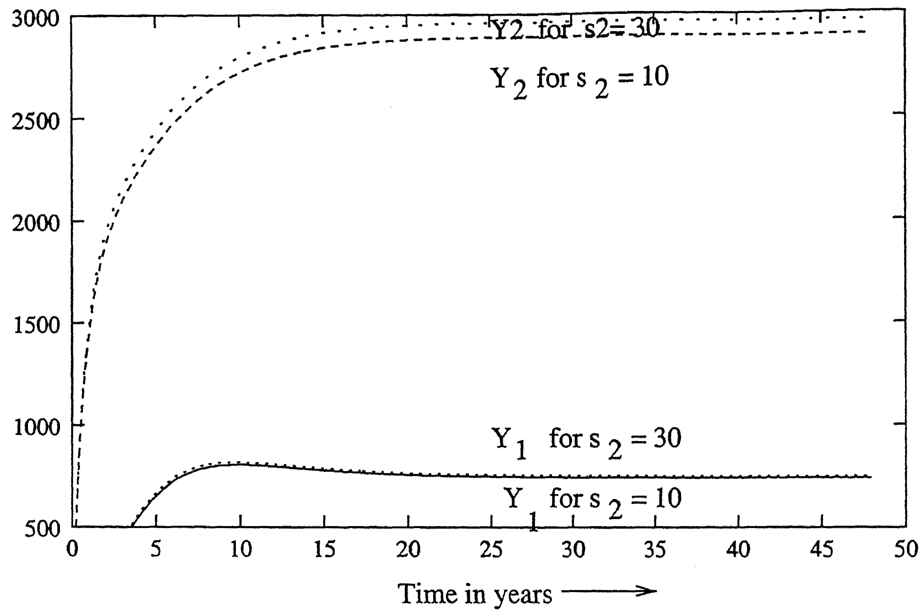


Figure 7.5: Variation of Y_1 and Y_2 with time for different rates of release of bacteria from the infective population.

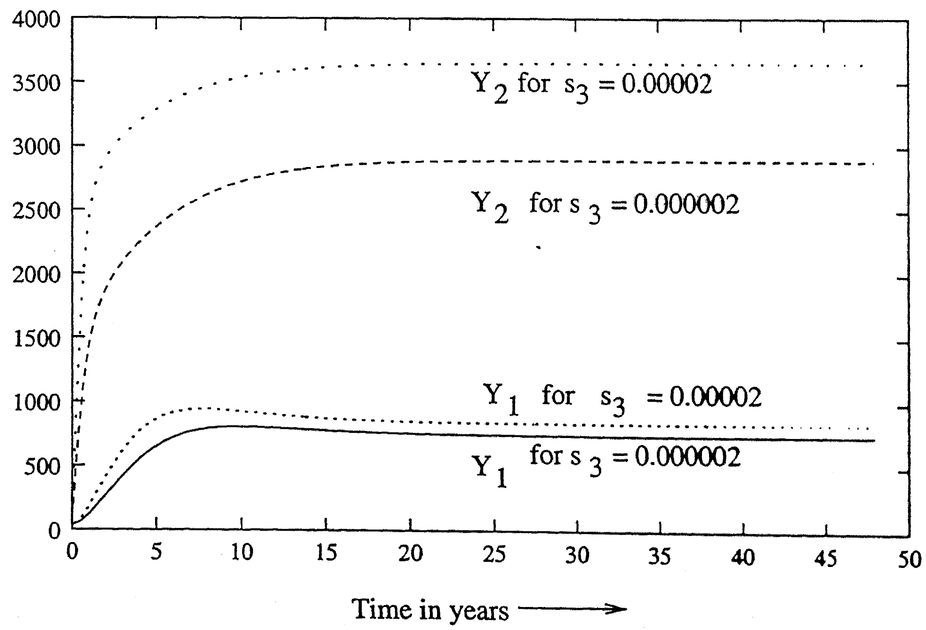


Figure 7.6: Variation of Y_1 and Y_2 with time for different rates of growth of bacteria population due to the environmental discharges.

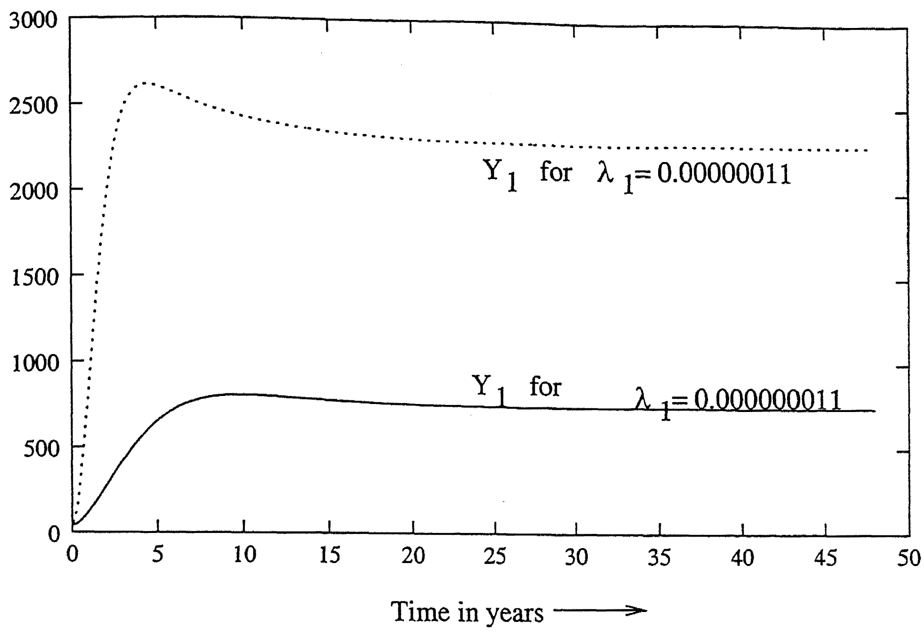


Figure 7.7: Variation of Y_1 with time for different disease transmission coefficients due to infectives of poor population.

these figures,

(i) if the bacteria population increases due to household discharges or otherwise, the spread of the infectious disease in both the rich and poor populations increases,

(ii) if infectives in poor population increase, the disease spreads faster in the rich population and

(iii) if the disease transmission coefficient between the infective poor population and the rich population increases then so does the number of infectives in the rich population. Therefore, it is in the interest of rich people not only to clean their own habitat but also the nearby habitat of poor people.

7.2.2 Case II: Q is a Function of N_2

As before, in this case also, it is sufficient to consider following subsystem of the original system of equation (7.1) with the same initial conditions,

$$\dot{Y}_1 = \beta_1(N_1 - Y_1)Y_1 + \lambda_1(N_1 - Y_1)Y_2 - (\nu_1 + \alpha_1 + d_1)Y_1,$$

$$
\begin{aligned}
\dot{N}_1 &= A_1 - d_1 N_1 - \alpha_1 Y_1,\\
\dot{Y}_2 &= \beta_2(N_2 - Y_2)Y_2 + \lambda_2(N_2 - Y_2)B_2 - (\nu_2 + \alpha_2 + d_2)Y_2,\\
\dot{N}_2 &= A_2 - d_2 N_2 - \alpha_2 Y_2, \qquad (7.10)\\
\dot{B}_2 &= sB_2\left(1 - \frac{B_2}{L}\right) + s_2 Y_2 - s_{20} B_2 + s_3 B_2 E,\\
\dot{E} &= Q_0 + lN_2 - \delta_0 E.
\end{aligned}
$$

The region of attraction T'

$$
\begin{aligned}
T' = &\left\{(Y_1, N_1, Y_2, N_2, B_2, E) : 0 \le Y_1 \le N_1 \le \frac{A_1}{d_1},\ 0 \le Y_2 \le \frac{A_2}{d_2},\ 0 \le B_2 \le B_{2\max},\right.\\
&\left. 0 \le E \le \frac{Q_0 + l\frac{A_2}{d_2}}{\delta_0}\right\},
\end{aligned}
$$

where $B_{2\max} = \frac{L}{2s}\left[\left\{s - s_{20} + s_3\frac{Q(\frac{A_2}{d_2})}{\delta_0}\right\} + \sqrt{\left\{s - s_{20} + s_3\frac{Q(\frac{A_2}{d_2})}{\delta_0}\right\}^2 + \frac{4ss_2A_2}{d_2L}}\right]$,
is positively invariant and all solutions starting in this region T' stay in T'. The continuity of the right hand side of (7.10) and its derivatives imply that a unique solution exists (Hale 1969).

The result of equilibrium analysis is stated in the following theorem.

THEOREM 7.3 There exist the following three equilibria, namely

(i) $E_1\left(0, \frac{A_1}{d_1}, 0, \frac{A_2}{d_2}, 0, \frac{Q_0 + l\frac{A_2}{d_2}}{\delta_0}\right)$,

(ii) $E_2\left(Y_1^*, N_1^*, 0, \frac{A_2}{d_2}, 0, \frac{Q_0 + l\frac{A_2}{d_2}}{\delta_0}\right)$, *which exists if* $\beta_1\frac{A_1}{d_1} > (\nu_1 + \alpha_1 + d_1)$,

where $Y_1^* = \frac{\left\{\beta_1\frac{A_1}{d_1} - (\nu_1 + \alpha_1 + d_1)\right\}d_1}{\beta_1(d_1 + \alpha_1)}$ *and* $N_1^* = \frac{\beta_1 A_1 + \alpha_1(\nu_1 + \alpha_1 + d_1)}{\beta_1(\alpha_1 + d_1)}$,

(iii) $E_3(\hat{Y}_1, \hat{N}_1, \hat{Y}_2, \hat{N}_2, \hat{B}_2, \hat{E})$, *which exists if* $s_2 > \frac{s_3 l \alpha_2 \bar{\bar{B}}_2}{\delta_0 d_2}$,

where $\bar{\bar{B}}_2 = \frac{L}{s}\{s - s_{20} + s_3\frac{Q_0}{\delta_0} + \frac{s_3 l A_2}{\delta_0 d_2}\}$.

Proof: Existence of either of the equilibria E_1 or E_2 is easy to show. Hence we only prove the existence of E_3 by the isocline method. Equating the right hand sides of the above equations in (7.10) to zero and after some manipulation we get

$$N_1 = \frac{A_1 - \alpha_1 Y_1}{d_1}, \quad N_2 = \frac{A_2 - \alpha_2 Y_2}{d_2}, \quad E = \frac{Q_0 + lN_2}{\delta_0}, \tag{7.11}$$

$$\beta_1\left(1+\frac{\alpha_1}{d_1}\right)Y_1^2 - \left\{\beta_1\frac{A_1}{d_1} - (\nu_1+\alpha_1+d_1)\right\}Y_1 + \lambda_1\left(1+\frac{\alpha_1}{d_1}\right)Y_1Y_2 - \lambda_1\frac{A_1}{d_1}Y_2 = 0, \tag{7.12}$$

$$\beta_2\left(1+\frac{\alpha_2}{d_2}\right)Y_2^2 - \left\{\frac{\beta_2 A_2}{d_2} - (\nu_2+\alpha_2+d_2) - \lambda_2\left(1+\frac{\alpha_2}{d_2}\right)B_2\right\}Y_2 - \lambda_2\frac{A_2}{d_2}B_2 = 0 \tag{7.13}$$

$$Y_2 = \frac{\left[\frac{s}{L}B_2^2 - \left(s - s_{20} + s_3\frac{Q_0}{\delta_0} + \frac{s_3 l}{\delta_0}\frac{A_2}{d_2}\right)B_2\right]}{s_2 - \frac{s_3 l\alpha_2}{\delta_0 d_2}B_2}. \tag{7.14}$$

We now show the existence of $\hat{Y}_2$ and $\hat{B}_2$ from (7.13) and (7.14). Then $\hat{Y}_1$, $\hat{N}_1$, $\hat{N}_2$ and $\hat{E}$ can be obtained from (7.11) and (7.12). The equation (7.13) is same as (7.7). From (7.14), we have,

(i) when $B_2 = 0$, $Y_2 = 0$,

(ii) when $Y_2 = 0$, $B_2 = 0$ or $B_2 = \frac{L}{s}\{s - s_{20} + s_3\frac{Q_0}{\delta_0} + \frac{s_3 l A_2}{\delta_0 d_2}\} = \bar{\bar{B}}_2$,

(iii) the slope of this curve is given by

$$\frac{dY_2}{dB_2} = \frac{\frac{2s}{L}B_2 - \left(s - s_{20} + s_3\frac{Q_0}{\delta_0} + \frac{s_3 l A_2}{\delta_0 d_2}\right) + \frac{s_3 l\alpha_2}{\delta_0 d_2}Y_2}{\left(s_2 - \frac{s_3 l\alpha_2}{\delta_0 d_2}B_2\right)},$$

(iv) at $(0,0)$, the slope is $-\dfrac{s - s_{20} + s_3\frac{Q_0}{\delta_0} + \frac{s_3 l A_2}{\delta_0 d_2}}{s_2} < 0$,

(v) at the point $(\bar{\bar{B}}_2, 0)$, the slope is given by $\dfrac{\frac{s}{L}\bar{\bar{B}}_2}{s_2 - \frac{s_3 l\alpha_2\bar{\bar{B}}_2}{\delta_0 d_2}}$, which is positive for $s_2 > \frac{s_3 l\alpha_2\bar{\bar{B}}_2}{\delta_0 d_2}$.

We make this assumption because for $l = 0$, we must get the slope at $(\bar{\bar{B}}_2, 0)$ positive as in Case I.

As before plotting curves (7.13) and (7.14), we get two intersecting points $(0,0)$ and $(\hat{B}_2, \hat{Y}_2)$. Also it can be checked that $\hat{Y}_1 < \frac{A_1}{d_1+\alpha_1}$ and $\hat{Y}_2 < \frac{A_2}{d_2+\alpha_2}$ and $\hat{B}_2$ increases with an increase in Q_0. Thus corresponding to $(\hat{B}_2, \hat{Y}_2)$, we have a third equilibrium point as $E_3(\hat{Y}_1, \hat{N}_1, \hat{Y}_2, \hat{N}_2, \hat{B}_2, \hat{E})$.

7.2.2.1 Stability Analysis

In the following we discuss the stability of the system (7.10). The local stability results of these equilibria are stated in the following theorem.

THEOREM 7.4 The equilibria E_1 and E_2 are unstable and the third equilibrium E_3 is locally asymptotically stable provided, $a_3(a_1a_2 - a_0a_3) - a_1^2 > 0$, where a_1, a_2 and a_3 are given explicitly in the proof of the theorem.

Proof: The variational matrix M corresponding to system (7.10) is

$$M = \begin{pmatrix} m_{11} & \beta_1 Y_1 + \lambda_1 Y_2 & \lambda_1(N_1 - Y_1) & 0 & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{33} & \beta_2 Y_2 + \lambda_2 B_2 & \lambda_2(N_2 - Y_2) & 0 \\ 0 & 0 & -\alpha_2 & -d_2 & 0 & 0 \\ 0 & 0 & s_2 & 0 & m_{55} & s_3 B_2 \\ 0 & 0 & 0 & l & 0 & -\delta_0 \end{pmatrix},$$

where

$m_{11} = \beta_1 N_1 - 2\beta_1 Y_1 - \lambda_1 Y_2 - (\nu_1 + \alpha_1 + d_1)$,
$m_{33} = \beta_2 N_2 - 2\beta_2 Y_2 - \lambda_2 B_2 - (\nu_2 + \alpha_2 + d_2)$ and
$m_{55} = s - s_{20} + s_3 E - \frac{2s}{L} B_2$.
The variational matrix M_1 at the equilibrium point E_1 is

$$M_1 = \begin{pmatrix} \beta_1 \frac{A_1}{d_1} - (\nu_1 + \alpha_1 + d_1) & 0 & \frac{\lambda_1 A_1}{d_1} & 0 & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_2 \frac{A_2}{d_2} - (\nu_2 + \alpha_2 + d_2) & 0 & \frac{\lambda_2 A_2}{d_2} & 0 \\ 0 & 0 & -\alpha_2 & -d_2 & 0 & 0 \\ 0 & 0 & s_2 & 0 & m_{55}^* & 0 \\ 0 & 0 & 0 & l & 0 & -\delta_0 \end{pmatrix},$$

where $m_{55}^* = s - s_{20} + s_3 \left(\dfrac{Q_0 + l\frac{A_2}{d_2}}{\delta_0} \right)$.
Now it is noted that at least one eigenvalue of M_1 is positive or has positive real part, implying instability of the equilibrium E_1.

The variational matrix M_2 at the equilibrium point E_2 is

$$M_2 = \begin{pmatrix} -\beta_1 Y_1^* & \beta_1 Y_1^* & \lambda_1 N_1^* & 0 & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_2 \frac{A_2}{d_2} - (\nu_2 + \alpha_2 + d_2) & 0 & \lambda_2 \frac{A_2}{d_2} & 0 \\ 0 & 0 & -\alpha_2 & -d_2 & 0 & 0 \\ 0 & 0 & s_2 & 0 & s - s_{20} + s_3 E^* & 0 \\ 0 & 0 & 0 & l & 0 & -\delta_0 \end{pmatrix},$$

where $E^* = \dfrac{Q_0 + l\frac{A_2}{d_2}}{\delta_0}$.

The matrix M_2 also has at least one positive root or root with positive real part. So the equilibrium E_2 is unstable.

The variational matrix M_3 at the equilibrium point E_3 is

$$M_3 = \begin{pmatrix} \hat{m_{11}} & \beta_1 \hat{Y_1} + \lambda_1 \hat{Y_2} & \lambda_1(\hat{N_1} - \hat{Y_1}) & 0 & 0 & 0 \\ -\alpha_1 & -d_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{m_{33}} & \beta_2 \hat{Y_2} + \lambda_2 \hat{B_2} & \lambda_2(\hat{N_2} - \hat{Y_2}) & 0 \\ 0 & 0 & -\alpha_2 & -d_2 & 0 & 0 \\ 0 & 0 & s_2 & 0 & \hat{m_{55}} & s_3 \hat{B_2} \\ 0 & 0 & 0 & l & 0 & -\delta_0 \end{pmatrix},$$

where $\hat{m_{11}} = -\left(\beta_1 \hat{Y_1} + \frac{\lambda_1 \hat{N_1} \hat{Y_2}}{\hat{Y_1}}\right)$, $\hat{m_{33}} = -(\beta_2 \hat{Y_2} + \frac{\lambda_2 \hat{N_2} \hat{B_2}}{\hat{Y_2}})$, $\hat{m_{55}} = -(\frac{s}{L}\hat{B_2} + s_2 \frac{\hat{Y_2}}{\hat{B_2}})$.

The characteristic polynomial corresponding to the above matrix is given by

$$\{\psi^2 + (\beta_1 \hat{Y_1} + \frac{\lambda_1 \hat{N_1} \hat{Y_2}}{\hat{Y_1}} + d_1)\psi + d_1(\beta_1 \hat{Y_1} + \frac{\lambda_1 \hat{N_1} \hat{Y_2}}{\hat{Y_1}}) + \alpha_1(\beta_1 \hat{Y_1} + \lambda_1 \hat{Y_2})\}$$
$$\times\{\psi^4 + a_3\psi^3 + a_2\psi^2 + a_1\psi + a_0\} = 0,$$

where

$$\begin{aligned}
a_3 &= \beta_2 \hat{Y_2} + \frac{\lambda_2 \hat{N_2} \hat{B_2}}{\hat{Y_2}} + d_2 + \delta_0 + \frac{s}{L}\hat{B_2} + s_2 \frac{\hat{Y_2}}{\hat{B_2}} \ > \ 0, \\
a_2 &= \left(\beta_2 \hat{Y_2} + \frac{\lambda_2 \hat{N_2} \hat{B_2}}{\hat{Y_2}}\right)(d_2 + \delta_0 + \frac{s}{L}\hat{B_2} + s_2 \frac{\hat{Y_2}}{\hat{B_2}}) + d_2\left(\delta_0 + \frac{s}{L}\hat{B_2} + s_2 \frac{\hat{Y_2}}{\hat{B_2}}\right) \\
&\quad (\frac{s}{L}\hat{B_2} + s_2 \frac{\hat{Y_2}}{\hat{B_2}})\delta_0 + \alpha_2(\beta_2 \hat{Y_2} + \lambda_2 \hat{B_2}) - s_2\lambda_2(\hat{N_2} - \hat{Y_2}) \ > \ 0, \\
a_1 &= d_2(\beta_2 \hat{Y_2} + \frac{\lambda_2 \hat{N_2} \hat{B_2}}{\hat{Y_2}})\left(\frac{s}{L}\hat{B_2} + s_2 \frac{\hat{Y_2}}{\hat{B_2}}\right) + d_2\delta_0\left(\frac{s}{L}\hat{B_2} + s_2 \frac{\hat{Y_2}}{\hat{B_2}}\right) \\
&\quad + \delta_0\left(\frac{s}{L}\hat{B_2} + s_2 \frac{\hat{Y_2}}{\hat{B_2}}\right)\left(\beta_2 \hat{Y_2} + \frac{\lambda_2 \hat{N_2} \hat{B_2}}{\hat{Y_2}}\right) + \delta_0 d_2\left(\beta_2 \hat{Y_2} + \frac{\lambda_2 \hat{N_2} \hat{B_2}}{\hat{Y_2}}\right) \\
&\quad + \alpha_2(\beta_2 \hat{Y_2} + \lambda_2 \hat{B_2})\{\delta_0 + \frac{s}{L}\hat{B_2} + s_2 \frac{\hat{Y_2}}{\hat{B_2}}\} - s_2\lambda_2(\hat{N_2} - \hat{Y_2})(d_2 + \delta_0) \ > \ 0,
\end{aligned}$$

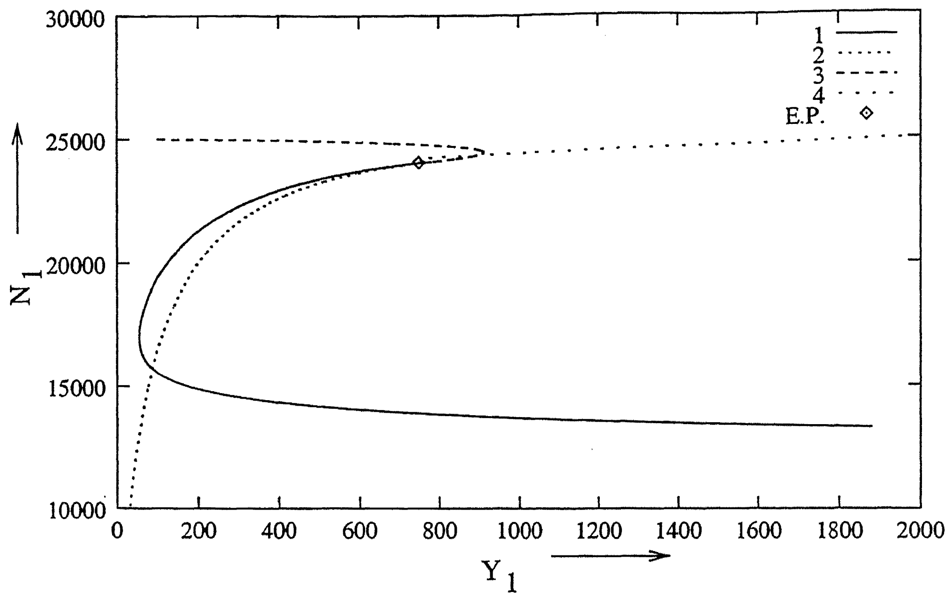


Figure 7.8: Variation of N_1 with Y_1.

$$\begin{aligned} a_0 &= d_2\delta_0\left(\beta_2\hat{Y}_2 + \frac{\lambda_2\hat{N}_2\hat{B}_2}{\hat{Y}_2}\right)\left(\frac{s}{L}\hat{B}_2 + s_2\frac{\hat{Y}_2}{\hat{B}_2}\right) + \alpha_2(\beta_2\hat{Y}_2 + \lambda_2\hat{B}_2)\delta_0\left(\frac{s}{L}\hat{B}_2 + s_2\frac{\hat{Y}_2}{\hat{B}_2}\right) \\ &\quad -\lambda_2(\hat{N}_2 - \hat{Y}_2)\{\alpha_2 l s_3\hat{B}_2 + s_2 d_2\delta_0\} > 0. \end{aligned}$$

By the Routh-Hurwitz criteria, E_3 is locally asymptotically stable if following conditions are satisfied.

$$a_3 > 0, \quad \begin{vmatrix} a_3 & a_1 \\ 1 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_3 & a_1 & 0 \\ 1 & a_2 & a_0 \\ 0 & a_3 & a_1 \end{vmatrix} > 0, \quad \begin{vmatrix} a_3 & a_1 & 0 & 0 \\ 1 & a_2 & a_0 & 0 \\ 0 & a_3 & a_1 & 0 \\ 0 & 1 & a_2 & a_0 \end{vmatrix} > 0.$$

The first two inequalities are obvious. If the third is satisfied, then so is the fourth one as $a_0 > 0$. Hence E_3 is locally asymptotically stable under the condition mentioned in the theorem.

Nonlinear Analysis and Simulation: As before it is speculated that system (7.10) may be globally stable, provided that we start away from other equilibria (see Appendix III). To show this, the system (7.10) is integrated using the fourth order Runge-Kutta method and using the same parameter values as in Case I with $Q_0 = Q_a$ and an additional parameter value $l = 0.0005$. The equilibrium values of $\hat{Y}_1$, $\hat{N}_1$, $\hat{Y}_2$, $\hat{N}_2$, $\hat{B}_2$ and $\hat{E}$ have been found as

$$\hat{Y}_1 = 757.59, \hat{N}_1 = 24053.01, \hat{Y}_2 = 3013.64, \hat{N}_2 = 21082.28, \hat{B}_2 = 2255803.38, \hat{E} = 30541.10.$$

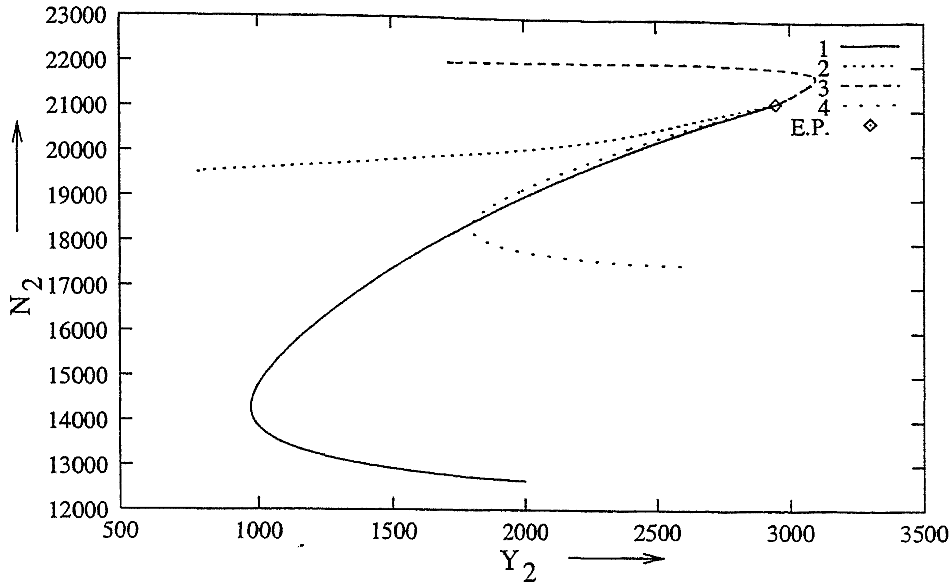


Figure 7.9: Variation of N_2 with Y_2.

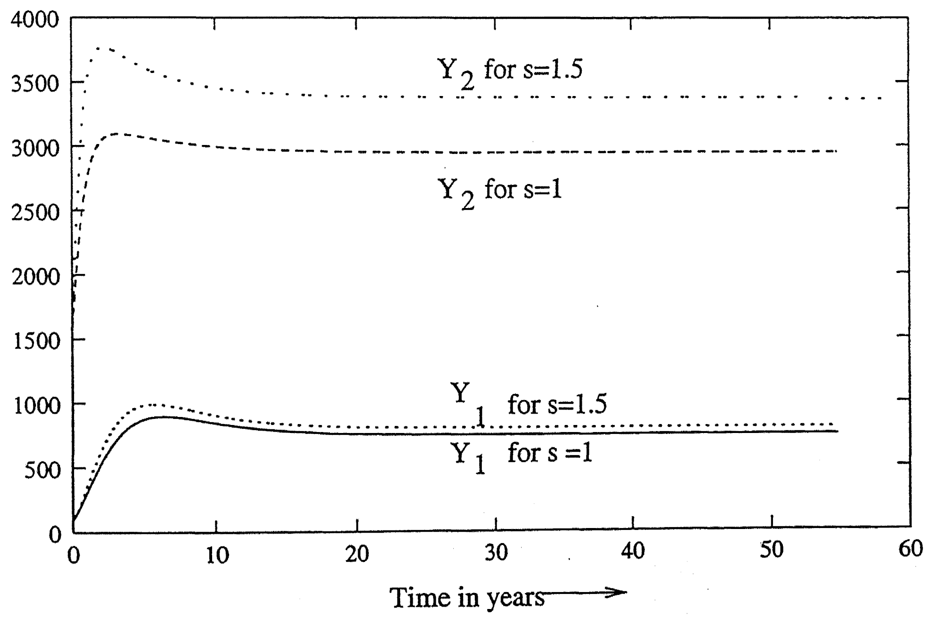


Figure 7.10: Variation of Y_1 and Y_2 with time for different intrinsic growth rates of the bacterial population.

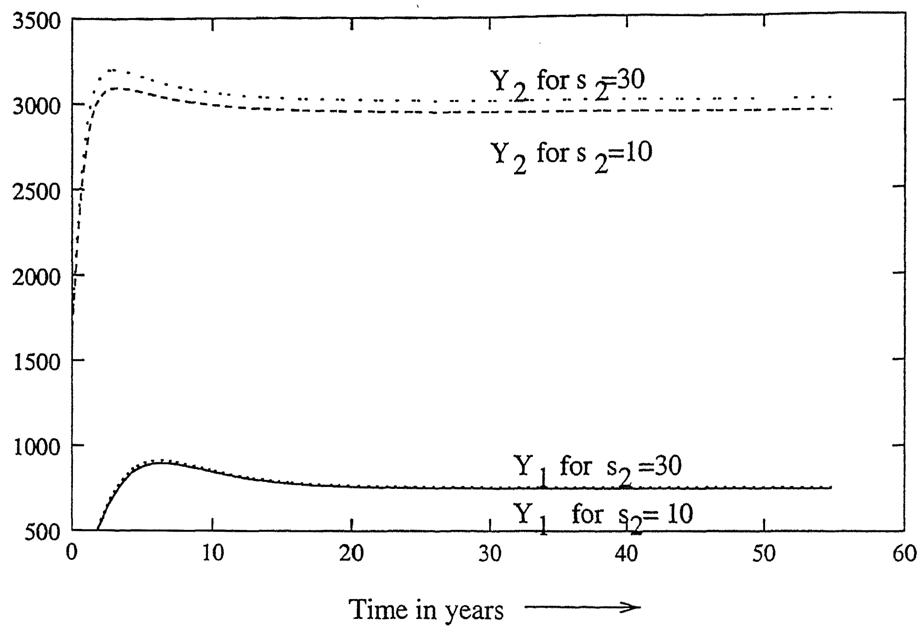


Figure 7.11: Variation of Y_1 and Y_2 with time for different rates of release of bacteria from the infective population.

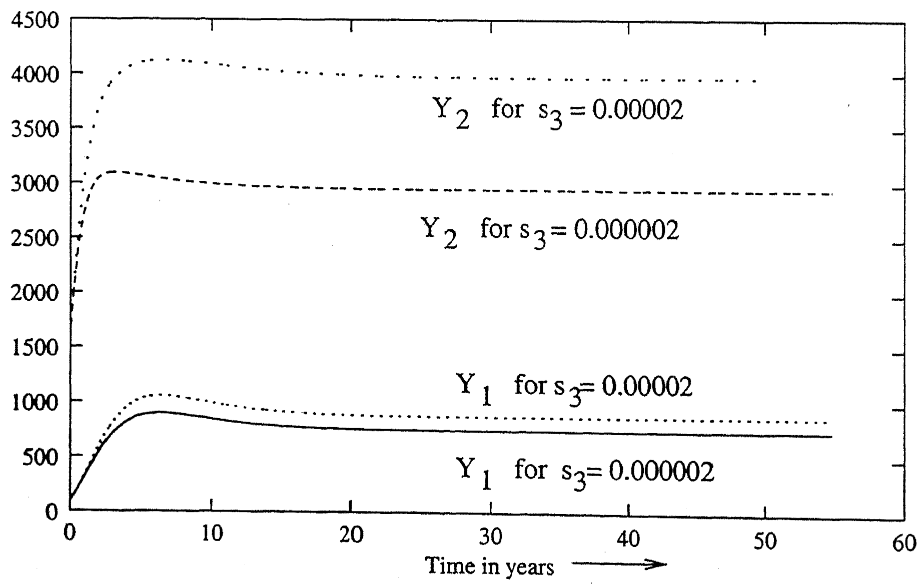


Figure 7.12: Variation of Y_1 and Y_2 with time for different rates of growth of bacteria population due to the environmental discharges.

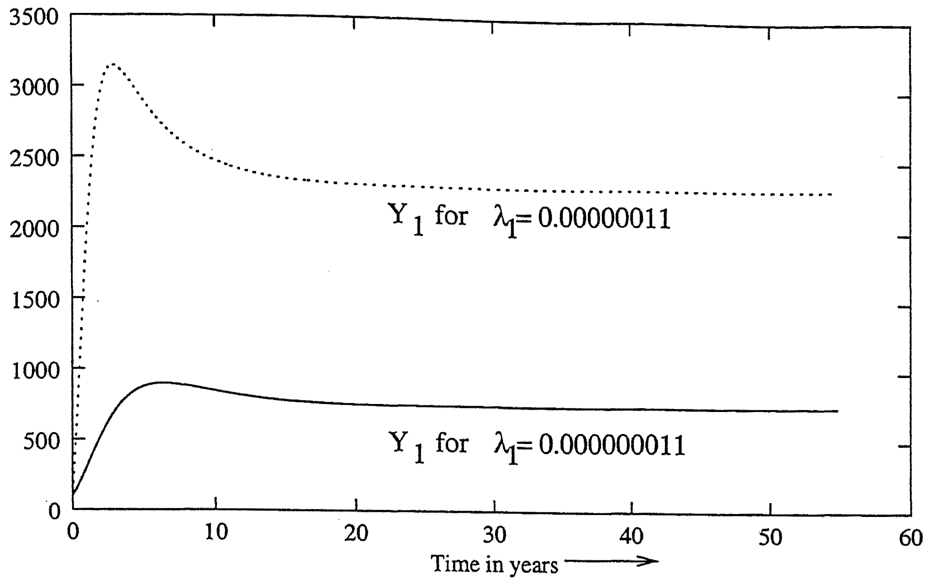


Figure 7.13: Variation of Y_1 with time for different disease transmission coefficients due to the infectives of the poor population.

In Figs. 7.9 and 7.10, we have plotted the infective population against the total population of respective classes for different initial positions 1, 2, 3 and 4. From the solution curves, we conclude that the system is globally stable about the endemic equilibrium point $(\hat{Y_1}, \hat{N_1}, \hat{Y_2}, \hat{N_2}, \hat{B_2}, \hat{E})$. The effects of various parameters on Y_1 and Y_2 are shown in Figs. 7.11-7.14 and similar results have been found as in Case I.

7.3 Conclusions

In this chapter, a nonlinear mathematical model is considered for infectious diseases caused by bacteria in a socially structured population (rich and poor) living in two nearby habitats, rich class living in a cleaner region and the poor class living in a neighboring region which is not so clean due to household discharges of poor people causing the growth of bacteria. Two cases are considered, (i) the rate of cumulative environmental discharges is a constant and (ii) the rate of cumulative environmental discharges is population dependent. In each case the existence of equilibria is shown and their local stability results are discussed. It is shown by simulation that the endemic equilibrium point in

each case is globally stable under its local stability conditions, provided we start away from other equilibria. By simulation, it is observed that if λ_1, i.e. the interaction between rich and poor people increases, the infective population of the rich class increases as expected. Also as other parameters such as the growth rate of the bacteria population or the rate of release of bacteria from infectives or the rate of growth of the bacteria population increases, infective populations of both the classes increase. This suggests that the spread of the infectious disease in rich people living in a better environment increases due to the interaction with service providers, who are living in relatively poor environmental conditions. This further suggests that rich people must involve themselves economically and otherwise to improve the habitat of service providers.

Bibliography

[1] Anderson, R.M., May, R.M., Population Biology of Infectious Diseases Part I *Nature vol. 280* p **361-367**, 1979.

[2] Anderson, R.M., May, R.M., The population dynamics of microparasites and their invertebrate host, *Trans. Roy. Phil. Soc. Series B, 291* p **451-524**, 1981.

[3] Bailey, N.T.J., The Mathematical Theory of Epidemics, *Griffin, London*, 1957.

[4] Bailey, N.T.J., The Mathematical Theory of Infectious Diseases and its Applications,*Griffin, London*, 1975.

[5] Bailey, N.T.J., Introduction to the modelling of venereal disease, *J. Math. Biology, 8* p **301-302** , 1979.

[6] Bailey, N.T.J., Spatial models in the epidemiology of infectious diseases, *Lecture Notes in Biomathematics, Springer, vol 38* p **233-261**, 1980.

[7] Bailey, N.T.J., The Biomathematics of Malaria, *Griffin, London*, 1982.

[8] Benenson, A.S., Control of communicable disease in man, *13 th ed. American P. H. Association, New York*, 1981.

[9] Brauer, F., Models for diseases with vertical transmission and nonlinear population dynamics, *Math. Biosci. 128* p **13-24**, 1995.

[10] Cairncross, S., Feachem, R. G., Environmental health engineering in the tropics, *John Wiley and Sons, New York*, 1983.

[11] Capasso, V., Global solution for a diffusive nonlinear deterministic epidemic model, *SIAM J. Appl. Math. 35* p **274-284**, 1978.

[12] Capasso, V., Serio, G., A generalization of the Kermack - McKendrick deterministic epidemic model, *Math. Biosci. 42* p **43-61**, 1978.

[13] Capasso, V., Maddalena, L., Convergence of equilibrium states for a reaction- diffusion system modelling the spatial spread of a class of bacterial and viral diseases, *J. Math. Bio. 13* p **175-184**, 1981.

[14] Cooke, K.L., Stability analysis for a vector disease model, *Rocky Mountain Journal of Mathematics, 9* p **31-42**, 1979.

[15] Cooke, K.L., Yorke, J. A., Some equations modelling growth processes and gonorrhea epidemics, *Math. Biosc. 16* p **75-101**, 1973.

[16] Das, S.C., Bhuyan, M., Baruah, L., Talukdar, P.K., Mosquito survey in Tripura, *Indian Journal of Malariology 28* p **129 -134**, 1991.

[17] de Jong, M.C.M., Diekmann, O., Heesterbeek, Hans, Epidemic Models : Their Structure and Relation to Data, edited by Denis Mollision *Cambridge University Press* p **85-95**, 1995.

[18] Dietz, K., Molineaux, L., Thomas, A., A Malaria model tested in the African Savannah, *Bull. Wld. Hlth. Org. 50*, p **347-357**, 1974.

[19] Dietz, K., Epidemiologicl interference of virus populations, *J. Math. Bio. 8,* p **291-300**, 1979.

[20] Dietz, K., The critical community size and eradication strategies for virus infections, *Lecture Notes, Autumn Course on Mathematical Ecology, ICTP, Trieste, Itly*, 1982.

[21] Dufour, A.P., Disease outbreak caused by drinking water, *J. WPCF vol. 54, 6* p **980-983**, 1982.

[22] Freedman, H.I., Deterministic mathematical models in population ecology, *HIFR Consulting Ltd., Edmonton*, 1987.

[23] Fuzzi, S., Mandrioli, P., Perfecto A., Fog droplets - An atmospheric source of secondary biological aerosol particles, *Atmospheric Environment vol. 31, no-2* p **287-290**, 1997.

[24] Gao, L.Q., Hethcote, H.W., Disease transmission models with density dependent demographics, *J. Math. Bio. 32* p **717-731**, 1992.

[25] Gao, L.Q., Meena-Lorca, J., Hethcote, H.W., Four SEI epidemic models with periodicity and separarices, *Math. Biosc. 128* p **157-184**, 1995.

[26] Greenhalgh, D., Some threshold and stability results for epidemic models with a density-dependent death rate, *Theor. Pop. Bio. 42* p **130-151**, 1990.

[27] Greenhalgh, D., Some results for an SEIR epidemic model with density dependence in the death rate, *IMA J. of Math. Applied in Medicine & Biology 9* p **67-106**, 1992.

[28] Greenhalgh, D., Das, R., Modelling epidemic with variable contact rates, *Theor. Pop. Bio. 47* p **129-179**, 1995.

[29] Gonzalez- Guzman, J., An epidemiological model for direct and indirect transmission of Typhoid Fever,*Math. Boisc., 96* p **33-46**, 1989.

[30] Hale, J.K., Ordinary Differential equations, New York:Wiley, 1969.

[31] Harold George Scott, Household and stored food insects of public health importance, U.S. Dept. of Health, Education and Welfare, Communicable Disease Center, Atlanta, Georgia, 1960.

[32] Harry, D.P., Kent, S.L., Ticks of public health importance and their control, U.S. Dept. of Health, Education and Welfare, Communicable Disease Center, Atlanta, Georgia, 1961.

[33] Harry, D.P., Kent, S.L., Lice of public health importance and their control, U.S. Dept. of Health, Education and Welfare, Communicable Disease Center, Atlanta, Georgia, 1961.

[34] Harry, D.P., John, S.W., Flea of public health importance and control, U.S. Dept. of Health, Education and Welfare, Communicable Disease Center, Atlanta, Georgia, 1962.

[35] Hethcote, H.W., Asymptotic behavior in a deterministic epidemic model, *Bull. Math. Bio. 35* p **607-614**, 1973.

[36] Hethcote, H.W., Asymptotic behavior and stability in epidemic models, *Lecture Notes in Biomathematics, vol. 2, Springer, New York* p **83-92**, 1974.

[37] Hethcote, H.W., Qualitative analysis of communicable disease models, *Math. Biosc. 28* p **335-356**, 1976.

[38] Hethcote, H.W., One thousand and one epidemic models, *In Frontiers in mathematical biology, S.A. Levin (Ed), Springer, New York*, 1994.

[39] Hethcote, H.W., Waltman, P., Optimal vaccination schedules in deterministic epidemic model, *Bull. Math. Bio. 35* p **365-381**, 1973.

[40] Hethcote, H.W., Stech, H.W., Van den Driessche, P., Periodicity and stability in epidemic models, *A survey in "Differential equations and applications in ecology, epidemics and pollution problems"*Edited by Busenberg, S.N. and Cooke, K. Academic Press, New York, 1981.

[41] Hethcote, H.W., Yorke, J.A., Gonorrhea transmission dynamics and control, *Lecture Notes in Biomathematics, Vol. 56, Springer, New York*, 1984.

[42] Hethcote, H.W., Van den Driessche, P., Some epidemiological models with nonlinear incidence, *J. Math. Bio. 29* p **271-287**, 1991.

[43] Hethcote, H.W., Van den Driessche, P., An SIS epidemic model with variable population size and a delay, *J. Math. Bio. 34* p **177-194**, 1995.

[44] Kamper, J.T., The effects of asymptotic attacks on the spread of infectious disease: A deterministic model, *Bull. Math. Bio., 40* **p 707-718**, 1978.

[45] Lakshmikantham, V., Leela, S., Differential and Integral Inequalities, *Academic Press*, 1969.

[46] La Salle, J., Lefschetz, S., Stability by Liapunov's Direct Method with Applications, *Academic Press, New York, London*, 1961.

[47] Lajmanovich, A., Yorke, J.A., A deterministic model for gonorrhea in a nonhomogeneous population, *Math. Biosc. 28*, **p 221-236**, 1976.

[48] Last, J.M., A dictionary of epidemiology, *Oxford University Press*, 1988.

[49] Liu, W., Levin, S.I., Iwasa, I., Influence of nonlinear incidence rates upon the behaviour of SIRS epidemiological models *J. Math. Bio. 23* **p 187-204**, 1986.

[50] Liu, W., Hethcote, H.W., Levin, S.I., Dynamical behaviour of epidemiological models with nonlinear incidence rates, *J. Math.Bio 25* **p 359-380**, 1987.

[51] Lotka, A.J., Contributions to the analysis of malaria epidemiology, *Am. J. Hyg. 3*, **p 1-121**, 1923.

[52] Lowe, C.R., Kostrzewski, J., Epidemiology: A guide to teaching methods, *Churchill Livingstone*, 1973.

[53] Ludwig, D., Final Size Distributions for Epidemics, *Math. Biosci., 23,* **p 33-46**, 1975.

[54] Macdonald, G., The analysis of Malaria epidemics *Tropical Disease Bull. 50***p 871-889**, 1953.

[55] Macdonald, G., The epidemiology and control of Malaria, *Oxford Press, London*, 1957.

[56] Manoharan, A., Jumblingam, A., Das, P.K., Utility of catalytic models in the estimation of incidence and prevalence of malaria in a hyperendemic situation, *South-East Aian J. Trop. Med. & Public Health 27*, p **738-741**, 1996.

[57] Marcati, P., Pozio, M.A., Global asymptotic stability for a vector disease model with spatial spread , *J. Math. Bio. 9* p **179-187**, 1980.

[58] May, R.M., Anderson R.M., Population Biology of Infectious Diseases Part II *Nature vol. 280* p **455-461**, 1979.

[59] Mena-Lorca, J., Hethcote, H.W., Dynamic models of infectious diseases as regulators of population size, *J. Math. Bio. 30* p **693-716**, 1992.

[60] Misra, O.P., Environmental effects on ecosystem: a contribution to mathematical ecology, *Ph.D. Thesis*, IIT Kanpur (INDIA), 1987.

[61] Molineaux, L., Dietz, K., Thomas A., Further epidemiological evaluation of a malaria model, *Bull. Wld. Hlth. Org. 56*, p **565-571**, 1978.

[62] Mollison, D., Spatial contact models for ecological and epidemic spread *J. Roy. Stat. Soc. B 39* p **283-326**, 1977.

[63] Mollison, D., Epidemic models: Their Structure and Relation to Data *Cambridge University Press* , 1995.

[64] Morse, S., Factors in the Emergence of Infectious Diseases, *EID, vol. 1, No. 1* p **7-15**, 1995.

[65] Murata, Y., Mathematics for Stability and Optimization of Economic Systems Academic Press, New York,, 1977.

[66] Nallaswamy, R., Shukla, J.B., Effects of dispersal on stability of gonorrhea endemic model, *Math. Biosc., 61* p **63-72**, 1982.

[67] Nchinda, T.C., Malaria: A re-emerging Disease in Africa, *Emerging Infectious Diseases*, vol. 4, No- **3p 398**, 1998.

[68] Park, K., Park's Textbook of Preventive and social medicine *M/s Banarsidas Bhanot publishers*, India, 1997.

[69] Purdom, P.W., Environmental Health, Academic Press, New York, 1980.

[70] Radcliffe, J., The initial geographical spread of host-vector and carrier borne epidemics *J. Appl. Prob. 10* p **703-717**, 1973.

[71] Radcliffe, J., The periodicity of endemic malaria, *J. Appl. Prob. 11, no 3,* p **562-567**, 1974.

[72] Ross, R., The amount of malaria depends on the number of the carriers, John Murray, London, 1929.

[73] Ross, R., The prevention of Malaria. Eng. Edn. John Murray, London, 1911.

[74] Sanchetz, D.A., Ordinary Differential Equations and Stability Theory: An Introduction, *Freeman, San Francisco*, 1968.

[75] Saxena, V.P., Juneja, A., Misra, O.P., Analysis of SIR epidemic model in a heterogeneous population, *J. Bio. System Vol. 1, No. 1* p **79-87**, 1993.

[76] Sharma, V.P., Re -emergence of malaria in India, *Indian J. Med. Research 103* p **26-45**, 1996.

[77] Sharma, V.P., Fighting malaria in India, *Current Science 75* p **1127-1140**, 1998.

[78] Shukla, J.B., Environmental effects on spread of infectious diseases : Some epidemic models, Paper discussed at *2nd Autumn course on mathematical ecology, ICTP Trieste, Italy Nov 10-Dec 12*, 1986.

[79] Shukla, J.B., Hallam, T.G., Capasso, V (Ed), Mathematical Modelling of Environmental and Ecological Systems, *Developments in Environmental Modelling, vol 11, Elsevier, Amsterdam*, 1987.

[80] Tapaswi, P.K., Ghosh, A.K., Mukhopadhyay, B.B., Transmission of Japanese Encephalitis in a 3-population model, *Ecological Modelling 83* p **295-309**, 1995.

[81] Taylor, I., Knowelden, J., Principles of Epidemiology, Little, Brown and Co., Boston, 1964.

[82] Trawis, C.C., Lenhart, S.M., Eradication of infectious diseases in heterogenous populations, *Math. Biosc. 83* p **191-198**, 1987.

[83] Waltman, P., Deterministic threshold models in the theory of epidemics, *Lecture Notes in Biomathematics, Vol. 1* Springer Verlag, New York, 1974.

[84] Webb, G.F., A reaction diffusive model for a deterministic diffusive epidemic, *J. Math. Anal. Appl. 84* p **150-161**, 1981.

[85] Wichmann, H.E., Asymptotic behaviour and stability in four models of venereal disease, *J. Math. Bio. 8* p **365-373**, 1979.

[86] Zhou,J., Hethcote, H.W, Population size dependent incidence in models for diseases without immunity, *J. Math. Bio. 32* p **809-304**, 1994.

Appendix I: Global stability proof of the system (5.2) for $\alpha_1 = 0$, $\alpha_2 = 0$

It may be pointed out here that the statement regarding the global stability of the non-trivial equilibrium point has been mentioned as a motivation for the numerical simulation carried out subsequently. To prove this, we reformulate the system by using the value of $Z_1 = \frac{d_1}{\delta_1} Y_1$ (obtained by putting $\dot{Z}_1 = 0$) in the rest of the equations of the system. Thus we consider the following reduced system of equations for global analysis:

$$\begin{aligned}
\dot{Y}_1 &= \beta_1 \left\{ N_1 - \left(1 + \frac{\delta_1}{d_1}\right) Y_1 \right\} Y_2 - (\nu_1 + d_1 + \delta_1) Y_1 \\
\dot{N}_1 &= A - d_1 N_1 \\
\dot{Y}_2 &= \beta_2 (\bar{N}_2 - Y_2) Y_1 + \lambda_2 (\bar{N}_2 - Y_2) \frac{\delta_1}{d_1} Y_1 - d_2 Y_2 \\
&= \left(\beta_2 + \lambda_2 \frac{\delta_1}{d_1}\right) \bar{N}_2 Y_1 - \left(\beta_2 + \lambda_2 \frac{\delta_1}{d_1}\right) Y_1 Y_2 - d_2 Y_2
\end{aligned} \tag{1}$$

where $\bar{d}_2 = d_2 + (1 - a') \frac{r_2}{K_2} \bar{N}_2$. The equilibrium point $(\hat{Y}_1, \hat{N}_1, \hat{Y}_2)$ of this system corresponds to the equilibrium point of the original system.

To check the stability of $(\hat{Y}_1, \hat{N}_1, \hat{Y}_2)$ we use the following Liapunov function:

$$V = \frac{1}{2}(Y_1 - \hat{Y}_1)^2 + \frac{k_1}{2}(N_1 - \hat{N}_1)^2 + \frac{k_2}{2}(Y_2 - \hat{Y}_2)^2.$$

Thus $\dot{V}$ using the above system (1) can be written as follows,

$$\begin{aligned}
\dot{V} = &-\left\{ \beta_1 (1 + \frac{\delta_1}{d_1}) Y_2 + \nu_1 + d_1 + \delta_1 \right\} (Y_1 - \hat{Y}_1)^2 - k_1 d_1 (N_1 - \hat{N}_1)^2 \\
&+ \left[\left\{ \beta_1 \hat{N}_1 - \beta_1 (1 + \frac{\delta_1}{d_1}) \hat{Y}_1 \right\} + k_2 \left\{ (\beta_2 + \lambda_2 \frac{\delta_1}{d_1})(\bar{N}_2 - \hat{Y}_2) \right\} \right] (Y_1 - \hat{Y}_1)(Y_2 - \hat{Y}_2) \\
&- k_2 \left\{ (\beta_2 + \lambda_2 \frac{\delta_1}{d_1}) Y_1 + d_2 \right\} (Y_2 - \hat{Y}_2)^2 + \beta_1 Y_2 (Y_1 - \hat{Y}_1)(N_1 - \hat{N}_1).
\end{aligned}$$

Using Sylvester criteria we get the following inequalities for negative definiteness of $\dot{V}$

$$4k_2\left(\nu_1+d_1+\delta_1\right)\left(d_2+\beta_2\lambda_2\frac{\delta_1}{d_1}Y_1\right) >$$

$$\left[\beta_1\left\{\hat{N}_1-\left(1+\frac{\delta_1}{d_1}\right)\hat{Y}_1\right\}+k_2\left\{(\beta_2+\lambda_2\frac{\delta_1}{d_1})(\bar{N}_2-\hat{Y}_2)\right\}\right]^2$$

and $\quad 4\beta_1 Y_2 k_1 d_1 > (\beta_1 Y_2)^2$.

From the equations for the equilibrium of the above system, we get

$$\left\{\beta_1\hat{N}_1-\beta_1(1+\frac{\delta_1}{d_1})\hat{Y}_1\right\}\left\{(\beta_2+\lambda_2\frac{\delta_1}{d_1})(\bar{N}_2-\hat{Y}_2)\right\}=(\nu_1+d_1+\delta_1)d_2.$$

Thus the first inequality reduces to

$$4(\nu_1+d_1+\delta_1)k_2\beta_2\lambda_2\frac{\delta_1}{d_1}Y_1 > \left[\left\{\beta_1\hat{N}_1-\beta_1(1+\frac{\delta_1}{d_1})\hat{Y}_1\right\}-k_2\left\{(\beta_2+\lambda_2\frac{\delta_1}{d_1})(\bar{N}_2-\hat{Y}_2)\right\}\right]^2.$$

Now choosing $k_2=\dfrac{\beta_1\hat{N}_1-\beta_1(1+\frac{\delta_1}{d_1})\hat{Y}_1}{(\beta_2+\lambda_2\frac{\delta_1}{d_1})(\bar{N}_2-\hat{Y}_2)}$, the last inequality gives $Y_1>0$.
Further taking the maximum of the right hand side of the second inequality gives

$$k_1 > \frac{\beta_1 Y_{2max}}{4}d_1.$$

Thus choosing k_1 and k_2 in the above manner, we get $\dot{V}$ negative definite implying global stability of the reformulated system in the interior of the region of attraction. Hence the global stability of the original system is expected.

As mentioned earlier the global stability is considered with respect to the interior of the region of attraction.

Appendix II: Global stability proof of the system (5.4) for $\alpha_1 = 0$

The proof of the global stability of the system (5.4) when $\alpha_1 = 0$ in the interior of the region of attraction is carried out for a reduced system only. By considering the case when $\alpha_1 = 0$, the system (5.4) can be written as

$$\begin{aligned}
\dot{Y_1} &= \beta_1(N_1 - Y_1 - Z_1)Y_2 - (\nu_1 + d_1 + \delta_1)Y_1 \\
\dot{Z_1} &= \delta_1 Y_1 - d_1 Z_1 \\
\dot{N_1} &= A - d_1 N_1 \\
\dot{Y_2} &= \beta_2(N_2 - Y_2)Y_1 + \lambda_2(N_2 - Y_2)Z_1 - \left\{\alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}N_2\right\}Y_2 \\
\dot{N_2} &= r_2 N_2\left(1 - \frac{N_2}{K_2}\right) - \alpha_2 N_2 + \delta_2 N_2 E \\
\dot{E} &= Q_0 + lN_1 - \delta_0 E
\end{aligned} \tag{1}$$

Using the asymptotic values N_1, N_2 and E respectively as follows:
$\bar{N_1} = \frac{A}{d_1}$, $\bar{N_2} = \frac{K_2}{r_2}\left(r_2 - \alpha_2 + \delta_2\bar{E}\right)$, $\bar{E} = \frac{1}{\delta_0}\left(Q_0 + \frac{Al}{d}\right)$,
and $Z_1 = \frac{d_1}{\delta_1}Y_1$ (obtained by putting $\dot{Z_1} = 0$), we get the following reduced two dimensional system:

$$\begin{aligned}
\dot{Y_1} &= \beta_1\left\{\bar{N_1} - \left(1 + \frac{\delta_1}{d_1}\right)Y_1\right\}Y_2 - (\nu_1 + d_1 + \delta_1)Y_1 \\
\dot{Y_2} &= \left(\beta_2 + \lambda_2\frac{\delta_1}{d_1}\right)(\bar{N_2} - Y_2)Y_1 - \bar{d_2}Y_2
\end{aligned} \tag{2}$$

where $\bar{d_2} = \alpha_2 + d_2 + (1-a')\frac{r_2}{K_2}\bar{N_2}$. Now it is easy to observe that the system (2) is globally stable by taking following Liapunov function:

$$V = \frac{1}{2}(Y_1 - \hat{Y_1})^2 + \frac{k_1}{2}(Y_2 - \hat{Y_2})^2,$$

where

$$k_1 = \frac{\left\{\beta_1\bar{N_1} - \beta_1\left(1 + \frac{\delta_1}{d_1}\right)\hat{Y_1}\right\}}{\left(\beta_2 + \lambda_2\frac{\delta_1}{d_1}\right)(\bar{N_2} - \hat{Y_2})},$$

and $\hat{Y_1}$ & $\hat{Y_2}$ are the equilibrium values corresponding to the system (2), which are the same as the nontrivial equilibrium values corresponding to the original system (1).

$$\Big[\dot{V} = -\left\{\beta_1\left(1 + \tfrac{\delta_1}{d_1}\right)Y_2 + \nu_1 + d_1 + \delta_1\right\}(Y_1 - \hat{Y_1})^2 + \Big[\left\{\beta_1\bar{N_1} - \beta_1\left(1 + \tfrac{\delta_1}{d_1}\right)\hat{Y_1}\right\}$$
$$+k_1\left(\beta_2 + \lambda_2\tfrac{\delta_1}{d_1}\right)(\bar{N_2} - \hat{Y_2})\Big](Y_1 - \hat{Y_1})(Y_2 - \hat{Y_2}) - k_1\left\{\left(\beta_2 + \lambda_2\tfrac{\delta_1}{d_1}\right)Y_1 + \bar{d_2}\right\}(Y_2 - \hat{Y_2})^2\Big].$$

Appendix III: Global stability proof of the systems (7.3) and (7.10) for $\alpha_1 = 0,\ \alpha_2 = 0$)

The system (7.3) is as follows:

$$\begin{aligned}
\dot{Y_1} &= \beta_1(N_1 - Y_1)Y_1 + \lambda_1(N_1 - Y_1)Y_2 - (\nu_1 + d_1)Y_1 \\
\dot{N_1} &= A_1 - d_1 N_1 \\
\dot{Y_2} &= \beta_2(N_2 - Y_2)Y_2 + \lambda_2(N_2 - Y_2)B_2 - (\nu_2 + d_2)Y_2 \\
\dot{N_2} &= A_2 - d_2 N_2 \\
\dot{B_2} &= sB_2\left(1 - \frac{B_2}{L}\right) + s_2 Y_2 - s_{20}B_2 + s_3\frac{Q_a}{\delta_0}B_2
\end{aligned} \tag{1}$$

Again using asymptotic values of N_1 and N_2 as $\frac{A_1}{d_1}$ and $\frac{A_2}{d_2}$ in the last system we get following reduced system as follows:

$$\begin{aligned}
\dot{Y_1} &= \beta_1\left(\frac{A_1}{d_1} - Y_1\right)Y_1 + \lambda_1\left(\frac{A_1}{d_1} - \dot{Y_1}\right)Y_2 - (\nu_1 + d_1)Y_1 \\
\dot{Y_2} &= \beta_2\left(\frac{A_2}{d_2} - Y_2\right)Y_2 + \lambda_2\left(\frac{A_2}{d_2} - Y_2\right)B_2 - (\nu_2 + d_2)Y_2 \\
\dot{B_2} &= sB_2\left(1 - \frac{B_2}{L}\right) + s_2 Y_2 - s_{20}B_2 + s_3\frac{Q_a}{\delta_0}B_2
\end{aligned} \tag{2}$$

Now taking the following Liapunov function corresponding to the system (2)

$$V = \frac{1}{2}(Y_1 - \hat{Y_1})^2 + \frac{k_1}{2}(Y_2 - \hat{Y_2})^2 + k_2\left\{B_2 - \hat{B_2} - \hat{B_2}ln\frac{B_2}{\hat{B_2}}\right\},$$

we get derivative of V as

$$\begin{aligned}
\dot{V} &= -\left[\beta_1 Y_1 + \lambda_1\frac{A_1}{d_1}\frac{\hat{Y_2}}{\hat{Y_1}}\right](Y_1 - \hat{Y_1})^2 + \lambda_1\left(\frac{A_1}{d_1} - Y_1\right)(Y_1 - \hat{Y_1})(Y_2 - \hat{Y_2}) \\
&\quad -k_1\left[\beta_2 Y_2 + \lambda_2\frac{A_2}{d_2}\frac{\hat{B_2}}{\hat{Y_2}}\right](Y_2 - \hat{Y_2})^2 + k_1\lambda_2\left(\frac{A_2}{d_2} - Y_2\right)(Y_2 - \hat{Y_2})(B_2 - \hat{B_2}) \\
&\quad -k_2\frac{s}{L}(B_2 - \hat{B_2})^2 - k_2\frac{s_2 Y_2}{B_2\hat{B_2}}(B_2 - \hat{B_2})^2 + k_2\frac{s_2}{\hat{B_2}}(B_2 - \hat{B_2})(Y_2 - \hat{Y_2})
\end{aligned}$$

Using Sylvester criteria, $\dot{V}$ will be negative definite if following inequalities are satisfied:

$$\text{(i)}\quad 4\lambda_1\frac{A_1}{d_1}\frac{\hat{Y_2}}{\hat{Y_1}}k_1\frac{1}{2}\frac{\lambda_2 A_2}{d_2}\frac{\hat{B_2}}{\hat{Y_2}} > \lambda_1^2\left(\frac{A_1}{d_1} - Y_1\right)^2,$$

$$\text{(ii)}\quad 4k_1\frac{1}{4}\frac{\lambda_2 A_2}{d_2}\frac{\hat{B}_2}{\hat{Y}_2}k_2\frac{1}{2}\frac{s}{L} > \left\{k_1\lambda_2\left(\frac{A_2}{d_2}-Y_2\right)\right\}^2,$$

$$\text{(iii)}\quad 4k_1\frac{1}{4}\frac{\lambda_2 A_2}{d_2}\frac{\hat{B}_2}{\hat{Y}_2}k_2\frac{1}{2}\frac{s}{L} > \left(\frac{k_2 s_2}{\hat{B}_2}\right)^2.$$

Taking maximum of right hand side of first two inequalities and simplifying all three inequalities, we get

$$k_1 > \frac{\lambda_1 A_1 \hat{Y}_1 d_2}{2d_1\lambda_2 A_2 \hat{B}_2}, \tag{3}$$

$$\frac{\hat{B}_2 s d_2}{2\hat{Y}_2 L\lambda_2 A_2} > \frac{k_1}{k_2}, \tag{4}$$

$$\frac{k_1}{k_2} > \frac{2d_2\hat{Y}_2 L s_2^2}{\lambda_2 A_2 \hat{B}_2^{\,3} s}. \tag{5}$$

We can choose k_1 satisfying (3). From (4) and (5) we get upper and lower limit of $\frac{k_1}{k_2}$, so if $\frac{\hat{B}_2^{\,2} s}{2\hat{Y}_2 L} > s_2$, the choice of k_2 is always possible such that (4) and (5) both are satisfied. Thus under above condition the system is globally stable.

Global stability proof of the system (7.10) with $\alpha_1 = 0$ and $\alpha_2 = 0$

In this case also taking asymptotic value of N_1, N_2 as before and asymptotic value of E as $\frac{Q_0 + l\frac{A_2}{d_2}}{\delta_0}$ we get system (2) with $\frac{Q_a}{\delta_0}$ replaced with $\frac{Q_0 + l\frac{A_2}{d_2}}{\delta_0}$. Thus similar argument is valid for the global stability of the system (7.10) with $\alpha_1 = 0$ and $\alpha_2 = 0$.